# The Dirac-Bergmann Algorithm and Gauge Fixing in Electrodynamics

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**Abstract**

In this document, I would like to demonstrate how the Dirac-Bergmann Algorithm (DBA) can be applied to derive the electromagnetic fields by via gauge fixing. In the process, I shall highlight that the observable quantities obtained from the DBA are all independent of the choice of gauge—these results will include non-integrable non-holonomic gauge choices.

## 1 Hamiltonian Formulation of Maxwell's Electrodynamics

Let us follow the route of Dirac's *Lectures on Quantum Mechanics*[1] by starting with the Lagrangian density describing Maxwell's electrodynamics

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{1}$$

where the field strength tensor given as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The conjugate momenta are found through

$$\Pi^\mu \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}. \tag{2}$$

Notice that (due to the anti-symmetry of $F^{\mu\nu}$) $\Pi^0 = 0$. Since $\Pi^0$ vanishes, the Lagrangian is singular; therefore, the Dirac-Bergmann Algorithm must be employed to systematically handle the constraints when transitioning to the Hamiltonian formalism. The constraint we have found on $\Pi^0$ is categorized as a *primary constraint*; we will use the notation $\phi_1 \equiv \Pi^0$ to denote the constraint throughout.

The canonical Hamiltonian density is found through a *Legendre transformation* of the Lagrangian density in eq. (1), *i.e.*

$$\mathcal{H}_c \equiv \Pi_\mu \partial_0 A^\mu - \mathscr{L} \tag{3a}$$

$$= \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i - A_0\partial_i\Pi^i + \text{ total derivatives} \tag{3b}$$

Observe that in the transition from the Lagrangian to the Hamiltonian formulation, we have broken the manifest Lorentz covariance by explicitly distinguishing the time coordinate from the spatial ones. With this in mind, all proceedings calculations will be taken at some fixed time $t = t_0$; we may define the Hamiltonian's and Lagrangian's as the integrals over spatial volumes of their corresponding densities.

For the sake of completeness, the canonical Hamiltonian is then

$$H_c = \int d^3x\, \frac{1}{4}F^{ij}(\vec{x})F_{ij}(\vec{x}) - \frac{1}{2}\Pi^i(\vec{x})\Pi^i(\vec{x}) - A_0(\vec{x})\partial_i\Pi^i(\vec{x}). \tag{4}$$

Ultimately, we are interested in $H_c$, thus we are free to drop the total derivatives from $\mathcal{H}_c$ in eq. (3b) and one may then redefine $\mathcal{H}_c$ as[1]

$$\mathcal{H}_c = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i - A_0\partial_i\Pi^i. \tag{5}$$

The primary constraint $\phi_1$ imposes a restriction on the phase-space of the theory. When the phase-space elements obey $\phi_1$, we say that they lie on the *constraint surface*. Now the canonical Hamiltonian $\mathcal{H}_c$ is only defined on the constraint surface; thus, we may arbitrarily extend $\mathcal{H}_c$ off the constraint surface as long as the extension vanishes on the constraint surface. Thus, we may adjoin the primary constraint weighted with a Lagrange multiplier to the canonical Hamiltonian; the Lagrange multiplier is to be treated as a dynamical but non-physical variable of the theory. We thus have

$$\mathcal{H}_p = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i - A_0\partial_i\Pi^i + \lambda_1\phi_1. \tag{6}$$

We may freely make use of Hamilton's principle of extremized action to derive the equations of motion for our system as follows:

$$\begin{aligned}
\delta S &= \int d^4x\, \delta(\Pi_\mu\partial_0 A^\mu) - \delta\mathcal{H}_p && \text{(7a)}\\
&= \int d^4x\, \delta\Pi_\mu\partial_0 A^\mu + \Pi_\mu\delta(\partial_0 A^\mu) - \frac{\partial\mathcal{H}_p}{\partial A^\mu}\delta A^\mu - \frac{\partial\mathcal{H}_p}{\partial\Pi^\mu}\delta\Pi^\mu - \frac{\partial\mathcal{H}_p}{\partial\lambda_1}\delta\lambda_1 && \text{(7b)}\\
&= \int d^4x\, \delta\Pi_\mu\partial_0 A^\mu - \overbrace{\partial_0\Pi_\mu\delta A^\mu}^{\text{TPR + IBP}} - \frac{\partial\mathcal{H}_p}{\partial A^\mu}\delta A^\mu - \frac{\partial\mathcal{H}_p}{\partial\Pi^\mu}\delta\Pi^\mu - \frac{\partial\mathcal{H}_p}{\partial\lambda_1}\delta\lambda_1 + \text{total derivatives} && \text{(7c)}\\
&= \int d^4x\, \left(\partial_0 A^\mu - \frac{\partial\mathcal{H}_p}{\partial\Pi^\mu}\right)\delta\Pi^\mu - \left(\partial_0\Pi^\mu + \frac{\partial\mathcal{H}_p}{\partial A^\mu}\right)\delta A^\mu - \frac{\partial\mathcal{H}_p}{\partial\lambda_1}\delta\lambda_1 && \text{(7d)}\\
&\overset{!}{=} 0. && \text{(7e)}
\end{aligned}$$

Notice that in eq. (7c) we used the *transposition rule*, which is valid as there are (at this point) no constraints on the $A^\mu$ fields. The variations of $\delta\Pi^\mu$ and $\delta A^\mu$ in eq. (7d) are arbitrary and independent, one finds the following equations of motion:

$$\dot{A}^\mu = \frac{\partial\mathcal{H}_p}{\partial\Pi^\mu}, \tag{8a}$$

$$\dot{\Pi}^\mu = -\frac{\partial\mathcal{H}_p}{\partial A^\mu}, \tag{8b}$$

$$\phi_1 = 0. \tag{8c}$$

By making use of eqs. (8a) and (8b), one can compute the time evolution of an arbitrary phase-space

[1] I will now stop distinguishing between the Hamiltonian and the Hamiltonian density.

function $F$ as

$$\dot{F}(\vec{x}) = \int d^3z \, \frac{\partial F(\vec{x})}{\partial A^\mu(\vec{z})} \dot{A}^\mu(\vec{z}) + \frac{\partial F(\vec{x})}{\partial \Pi^\mu(\vec{z})} \dot{\Pi}^\mu(\vec{z}) \tag{9a}$$

$$= \int d^3z \, \frac{\partial F(\vec{x})}{\partial A^\mu(\vec{z})} \frac{\partial \mathcal{H}_p}{\partial \Pi^\mu}(\vec{z}) - \frac{\partial F(\vec{x})}{\partial \Pi^\mu(\vec{z})} \frac{\partial \mathcal{H}_p}{\partial A^\mu}(\vec{z}) \tag{9b}$$

$$= \{F(\vec{x}), H_p\}, \tag{9c}$$

where the Poisson bracket $\{.\}$ of the arbitrary phase-space functions $F$ and $G$ is defined as

$$\{F(\vec{x}), G(\vec{y})\} \equiv \int d^3z \, \left( \frac{\partial F(\vec{x})}{\partial A^\mu(\vec{z})} \frac{\partial G(\vec{y})}{\partial \Pi_\mu(\vec{z})} - \frac{\partial G(\vec{y})}{\partial A^\mu(\vec{z})} \frac{\partial F(\vec{x})}{\partial \Pi_\mu(\vec{z})} \right). \tag{10}$$

Now, for our Hamiltonian theory to be consistent, we require that time evolution should not move us off the constrained phase-space. Thus, we must impose the condition that the time derivatives of all primary constraints should vanish. This requirement will in general, yield further constraints on the phase-space and potentially the Lagrange multipliers. If we find further constraints on the phase-space, we will categorize these constraints as *secondary constraints*. Accordingly, we compute $\dot{\phi}_1$ and demand that it vanishes; this procedure yields the following:

$$0 \overset{!}{=} \{\phi_1(\vec{x}), H_p\} \tag{11a}$$

$$= \int d^3z \, \left( \frac{\partial \phi_1(\vec{x})}{\partial A^\mu(\vec{z})} \frac{\partial \mathcal{H}_p}{\partial \Pi_\mu}(\vec{z}) - \frac{\partial \phi_1(\vec{x})}{\partial \Pi_\mu(\vec{z})} \frac{\partial \mathcal{H}_p}{\partial A^\mu}(\vec{z}) \right) \tag{11b}$$

$$= -\int d^3z \, \frac{\partial \mathcal{H}_p}{\partial A^\mu}(\vec{z}) g^{\mu 0} \delta^3(\vec{x} - \vec{z}) \tag{11c}$$

$$= -\frac{\partial \mathcal{H}_p}{\partial A^0}(\vec{x}) \tag{11d}$$

$$= \partial_i \Pi^i(\vec{x}) \tag{11e}$$

$$\implies 0 \overset{!}{=} \partial_i \Pi^i \tag{11f}$$

Thus, one finds a secondary constraint $\phi_2 \equiv \partial_i \Pi^i$. This process of applying consistency conditions on the constraints should be repeated; however the condition that $\dot{\phi}_2 = 0$, will yield the trivial solution $0 = 0$. To see this, we compute:

$$0 \overset{!}{=} \{\phi_2(\vec{x}), H_p\} \tag{12a}$$

$$= \int d^3z \, \left( \frac{\partial \phi_2(\vec{x})}{\partial A^\mu(\vec{z})} \frac{\partial \mathcal{H}_p}{\partial \Pi_\mu}(\vec{z}) - \frac{\partial \phi_2(\vec{x})}{\partial \Pi_\mu(\vec{z})} \frac{\partial \mathcal{H}_p}{\partial A^\mu}(\vec{z}) \right) \tag{12b}$$

$$= -\partial_i \int d^3z \, \overbrace{\frac{\partial \Pi^i(\vec{x})}{\partial \Pi_\mu(\vec{z})}}^{\eta^{i\mu}} \overbrace{\frac{\partial \mathcal{H}_p}{\partial A^\mu}}^{\eta^0{}_\mu}(\vec{z}) \tag{12c}$$

$$= 0 \tag{12d}$$

$$\tag{12e}$$

Hence, there are no tertiary constraints or any constraints on the Lagrange multipliers. As there are no constraints on the Lagrange multipliers, the total Hamiltonian ($\mathcal{H}_T$) and the primary Hamiltonian are equivalent.

Let us gather our results thus far:

$$\mathcal{H}_T = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i - A_0\partial_i\Pi^i + \lambda_1\phi_1, \tag{13}$$

$$\phi_1 = \Pi^0, \tag{14}$$

$$\phi_2 = \partial_i\Pi^i. \tag{15}$$

One may notice that all constraints at this point are *first class*, that is $\{\phi_i, \phi_j\} = 0$ for $i, j = 1, 2$.

The next step in the DBA is to remove all primary first class constraints from the total Hamiltonian, to then define the first class Hamiltonian

$$\mathcal{H}_{fc} \equiv \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i - A_0\partial_i\Pi^i. \tag{16}$$

One can check that $\{H_{fc}, \phi_j\} = 0$ for $j = 1, 2$, and thus the first class Hamiltonian is—as the name suggests—first class.

Keeping in line with the steps of the DBA, we should define the extended Hamiltonian by adjoining all independent first class constraints (primary, secondary, *etc.*) weighted by Lagrange multipliers to the first class Hamiltonian. We do this to account for the additional gauge symmetry which occurs in our theory that, through the *Dirac Conjecture*, arises from all first class constraints being the generators of gauge transformations. The extended Hamiltonian is then

$$\mathcal{H}_E = \mathcal{H}_{fc} + \Lambda_1\phi_1 + \Lambda_2\phi_2 \tag{17a}$$

$$= \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i - A_0\partial_i\Pi^i + \Lambda_1\phi_1 + \Lambda_2\phi_2. \tag{17b}$$

Let us now consider the dynamics of the $A^0$ phase-space variable. Computing $\dot{A}^0$, one finds the following:

$$\dot{A}^0(\vec{x}) = \{A^0(\vec{x}), H_E\} \tag{18a}$$

$$= \int d^3z \, \left( \frac{\partial A^0(\vec{x})}{\partial A^\mu(\vec{z})} \frac{\partial \mathcal{H}_E}{\partial \Pi_\mu(\vec{z})} - \frac{\partial \mathcal{H}_E}{\partial A^\mu(\vec{z})} \frac{\partial A^0(\vec{x})}{\partial \Pi_\mu(\vec{z})} \right) \tag{18b}$$

$$= \int d^3z \, g^{0\mu}\delta^3(\vec{x} - \vec{z}) \frac{\partial \mathcal{H}_E}{\partial \Pi_\mu(\vec{z})} \tag{18c}$$

$$= \frac{\partial \mathcal{H}_E}{\partial \Pi_0(\vec{x})} \tag{18d}$$

$$= \Lambda_1(\vec{x}) \tag{18e}$$

We thus find that temporal dynamics of $A^0$ are completely determined by the Lagrange multiplier $\Lambda_1$, which is an unphysical quantity. Recall that we found constraint $\Pi^0 = 0$; hence the canonically conjugate pair of variables $(A^0, \Pi^0)$, in the words of Dirac, *"are therefore not of interest at all [, and] we can drop them out from the theory, leading to a simplified Hamiltonian formalism where we have fewer degrees of freedom, but still retain all the degrees of freedom which are physically of interest"*. Now, under this *reduced phase-space*, the extended Hamiltonian has the form[2]

$$\mathcal{H}_E = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i + \Lambda\phi_2. \tag{19}$$

## 2 Gauge Fixing

In the reduced phase-space there is one constraint of interest, namely $\phi_2 = \nabla \cdot \vec{\Pi}$. Recall that $\phi_2$ was first class, and thus by Dirac's conjecture should be a generator of a gauge transformation. Notice that the Lagrange multiplier $\Lambda$ in eq. (19) is arbitrary, this arbitrariness is a non-physical component of the Hamiltonian theory. One may consider the evolution of a phase-space function $F$ and investigate how it depends on the arbitrary parameter of $\Lambda$; this can be done by evolving $F$ infinitesimally with the arbitrary parameter $\Lambda$ and then comparing this to the evolution with another parameter $\tilde{\Lambda}$. We define the transformation $\delta F$ as this difference of these two results; we compute this quantity explicitly:

$$\begin{aligned}
\delta F &= \dot{F}[\Lambda]\delta t - \dot{F}[\tilde{\Lambda}]\delta t && \text{(20a)}\\
&= \{F, H_E[\Lambda]\} - \{F, H_E[\tilde{\Lambda}]\} && \text{(20b)}\\
&= \int d^3x \, \{F, \big(\Lambda(\vec{x}) - \tilde{\Lambda}(\vec{x})\big)\phi_2(\vec{x})\}\delta t && \text{(20c)}\\
&= \int d^3x \, \overbrace{(\Lambda(\vec{x}) - \tilde{\Lambda}(\vec{x}))\delta t}^{\equiv \delta\Lambda}\{F, \phi_2(\vec{x})\} && \text{(20d)}\\
&= \int d^3x \, \delta\Lambda\{F, \phi_2(\vec{x})\} && \text{(20e)}
\end{aligned}$$

One may then build a finite gauge transformation through many of these infinitesimal gauge transformations, the finite transformation will be of the form

$$\delta F = \int d^3x \, \alpha(\vec{x})\{F, \phi_2(\vec{x})\}, \tag{21}$$

and we shall call $\alpha(\vec{x})$ a gauge parameter.

Let us investigate how the phase-space variables behave under this gauge transformation. The momenta are clearly invariant, *i.e.*

$$\delta\Pi^i = 0. \tag{22}$$

We shall call quantities that are gauge invariant *observables*; here we will denote the space of all observables as $\mathcal{O}$. As we will see, one can identify the momenta with components of the electric field, that is

$$\Pi^i = E^i. \tag{23}$$

[2] In the reduced phase-space, all indices now run over 1,2,3. The Euclidean metric is now used as opposed to the Minkowski metric.

The gauge transformation of the coordinate $A^i$ is

$$\delta A^i(\vec{x}) = \int d^3y\ \alpha(\vec{y})\{A^i(\vec{x}), \phi_2(\vec{y})\} \tag{24a}$$

$$= \int d^3y\ \alpha(\vec{y})\partial^j\{A^i(\vec{x}), \Pi^j(\vec{y})\} \tag{24b}$$

$$\stackrel{\text{IBP}}{=} -\int d^3y\ \delta^3(\vec{x}-\vec{y})\partial^i\alpha(\vec{y}) \tag{24c}$$

$$= -\partial^i\alpha(\vec{x}) \tag{24d}$$

One can check that $F^{ij}F_{ij}$ is invariant under the transformation

$$A^i(\vec{x}) \mapsto A^i(\vec{x}) - \partial^i\alpha(\vec{x}), \tag{25}$$

and thus so is $\mathcal{H}_c$. Hence, we find (as expected from the Dirac Conjecture) gauge symmetry within the Hamiltonian theory. The gauge symmetry is, of course, not a physical symmetry. Rather the gauge symmetry reflects a redundancy in our mathematical description of the dynamics of the theory. This redundancy invites us to further reduce the phase-space so that the mapping from the reduced phase space to $\mathcal{O}$ is bijective. We can achieve this further reduction in phase-space by *gauge fixing*. Gauge fixing involves defining a new constraint on the phase-space such that each *gauge orbit* intersects the constraint surface once and only once.

Notice that the gauge symmetry only lies within the coordinates $A^i$ and not within the momenta $\Pi^i$. Thus, to fix the gauge one should impose a constraint on only the coordinates and not the momenta. In anticipation of what will follow, we modify our notation slightly. We shall denote the secondary constraint $\phi_2$ as $C_1 = C_1[\Pi^i]$, and we shall denote the gauge fixing constraint[3] as $C_2 = C_2[A^i, \partial^j A^i]$. The constraint matrix defined as

$$C_{ab}(\vec{x},\vec{y}) \equiv \{C_a(\vec{x}), C_b(\vec{y})\}, \tag{26}$$

and is assumed to be non-singular. The non-singularity assumption will necessarily imply that the constraints $C_1$ and $C_2$ are *second class*.

Dirac has provided a very convenient method for dealing with second class constraints, we are instructed to *strongly* set all second class constraints to zero and then proceed with our calculations by using the *Dirac Bracket* which is defined as

$$\{F(\vec{x}), G(\vec{y})\}_D = \{F(\vec{x}), G(\vec{y})\} - \int d^3z_1 d^3z_2 \{F(\vec{x}), C_a(\vec{z}_1)\}C^{ab}(\vec{z}_1,\vec{z}_2)\{C_b(\vec{z}_2), G(\vec{y})\} \tag{27}$$

where $C^{ab}$ is the inverse to the constraint matrix $C_{ab}$. Since all constraints are now strongly set to zero and provided we use the Dirac bracket, we may use the following Hamiltonian in all calculations that follow

$$\mathcal{H} = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i. \tag{28}$$

[3] We include the possibility of non-holonomic and non-integrable constraints.

## 2.1 Coulomb Gauge

In this section we shall apply this formalism to the Coulomb gauge. This section has been adapted from[2]. The Coulomb gauge is defined by imposing the following constraint on the fields

$$C_2 \equiv \nabla \cdot \vec{A} = 0. \tag{29}$$

The Poisson bracket of the constraints in the Coulomb gauge is computed as follows:

$$\begin{aligned}
\{C_1(\vec{x}), C_2(\vec{y})\} &= \{\nabla_x^j \Pi^j(\vec{x}), \nabla_y^i A^i(\vec{y})\} && \text{(30a)}\\
&= \nabla_x^j \nabla_y^i \{\Pi^j(\vec{x}), A^i(\vec{y})\} && \text{(30b)}\\
&= -\nabla_x^j \nabla_y^i \delta^{ji} \delta^3(\vec{x}-\vec{y}) && \text{(30c)}\\
&= \nabla_x^i \nabla_x^i \delta^3(\vec{x}-\vec{y}) && \text{(30d)}\\
&= \nabla_x^2 \delta^3(\vec{x}-\vec{y}) && \text{(30e)}
\end{aligned}$$

Equation (30) allows us to compute the constraint matrix as:

$$\begin{aligned}
C_{ab}(\vec{x},\vec{y}) &= \{C_a(\vec{x}), C_b(\vec{y})\} && \text{(31a)}\\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab} \nabla_x^2 \delta^3(\vec{x}-\vec{y}), && \text{(31b)}
\end{aligned}$$

which can be formally inverted to obtain

$$C^{ab}(\vec{x},\vec{y}) = \delta^3(\vec{x}-\vec{y}) \frac{1}{\nabla_x^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ab}. \tag{32}$$

In what follows, we shall concern ourselves with the temporal dynamics of the phase-space variables $A^i$ and $\Pi^i$. The conjugate momenta are the simpler of the two, so we will begin there,

$$\begin{aligned}
\dot{\Pi}^i &= \{\Pi^i(\vec{x}), H\}_D && \text{(33a)}\\
&= \{\Pi^i(\vec{x}), H\} - \int d^3z_1 d^3z_2 \{\Pi^i(\vec{x}), C_a(\vec{z}_1)\} C^{ab}(\vec{z}_1,\vec{z}_2) \{C_b(\vec{z}_2), H\}. && \text{(33b)}
\end{aligned}$$

To compute $\dot{\Pi}^i$, one will need the following intermediate results:

$$\begin{aligned}
\{\Pi^i, H\} &= \nabla^2 A^i - \nabla^i \nabla \cdot \vec{A}, && \text{(34a)}\\
\{\Pi^i(\vec{x}), C_1(\vec{z}_1)\} &= 0, && \text{(34b)}\\
\{C_1(\vec{z}_2), H\} &= 0. && \text{(34c)}
\end{aligned}$$

These then imply:

$$\begin{aligned}
\dot{\Pi}^i &= \{\Pi^i(\vec{x}), H\}_D && \text{(35a)}\\
&= \{\Pi^i, H\} + 0 && \text{(35b)}\\
&= \nabla^2 A^i - \nabla^i \nabla \cdot \vec{A}. && \text{(35c)}
\end{aligned}$$

Now, to compute $\dot{A}^i$, one will need the following intermediate results:

$$\{A^i(\vec{x}), H\} = \Pi^i(\vec{x}), \tag{36a}$$
$$\{A^i(\vec{x}), C_1(\vec{y})\} = \nabla^i_y \delta^3(\vec{x} - \vec{y}), \tag{36b}$$
$$\{C_1(\vec{y}), H\} = 0, \tag{36c}$$
$$\{C_2(\vec{y}), H\} = \nabla^i \Pi^i(\vec{y}). \tag{36d}$$

With these, one then finds:

$$\dot{A}^i(\vec{x}) = \{A^i(\vec{x}), H\}_D \tag{37a}$$
$$= \{A^i(\vec{x}), H\} - \int d^3z_1 d^3z_2 \{A^i(\vec{x}), C_a(\vec{z}_1)\} C^{ab}(\vec{z}_1, \vec{z}_2)\{C_b(\vec{z}_2), H\} \tag{37b}$$
$$= \Pi^i - \nabla^i \frac{1}{\nabla^2} \nabla \cdot \vec{\Pi}. \tag{37c}$$

Gathering our calculations, we have:

$$\dot{A}^i = \Pi^i - \nabla^i \frac{1}{\nabla^2} \nabla \cdot \vec{\Pi}, \tag{38a}$$
$$\dot{\Pi}^i = \nabla^2 A^i - \nabla^i \nabla \cdot \vec{A}. \tag{38b}$$

These equations posses the same information as Maxwell's equations for the electric and magnetic fields. To see this, we just need to massage the equations in eq. (37) into a more recognizable form and identify $E^i = \Pi^i$ and $B^i = 1/2\epsilon^{ijk}F^{jk}$. We can then rewrite eq. (38b) as:

$$\partial_t E^i = \partial^j \partial^j A^i - \partial^i \partial^j A^j \tag{39a}$$
$$= \partial^j (\partial^j A^i - \partial^i A^j) \tag{39b}$$
$$= \partial^j F^{ji} \tag{39c}$$
$$= \partial^j \epsilon^{kji} B^k \tag{39d}$$
$$= (\nabla \times \vec{B})^i \tag{39e}$$
$$\implies \partial_t \vec{E} = \nabla \times \vec{B}. \tag{39f}$$

We may now take the curl of eq. (38a) to find:

$$\epsilon^{kji}\partial^k \dot{A}^i = \epsilon^{kji}\partial^k \Pi^i + \overbrace{\epsilon^{kji}\partial^k \partial^i}^{\text{0 by anti-symmetry}} \frac{1}{\nabla^2} \nabla \cdot \vec{\Pi} \tag{40a}$$
$$\partial_t \epsilon^{kji}\partial^k A^i = \epsilon^{kji}\partial^k E^i \tag{40b}$$
$$\partial_t (\nabla \times \vec{A}) = (\nabla \times \vec{E})^k \tag{40c}$$
$$\implies \partial_t \vec{B} = \nabla \times \vec{E} \tag{40d}$$

We thus recover Maxwell's equations of motion for the electric and magnetic fields. Notice that the constraint $\partial^i \Pi^i = 0$ is still true, so one still has that

$$\nabla \cdot \vec{E} = 0. \tag{41}$$

And of course, the statement that there are no magnetic monopoles

$$\nabla \cdot \vec{B} = 0, \tag{42}$$

is still true as this is a consequence of the symmetry of space-time and has nothing to do with dynamics of the Hamiltonian or Lagrangian systems. Equation (42) can be found by noting that

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{43}$$

where $\tilde{F}^{\mu\nu}$ is the Hodge dual of the electromagnetic field strength tensor $F^{\mu\nu}$. Equation (42) then directly follows from eq. (43).

## 2.2 Gauge Invariance of the Dirac-Bergmann Algorithm

In this section I would like to demonstrate how the results we derived in eq. (39f) and eq. (40d) are in fact quite independent of the gauge choice we make. To begin, one may recall that the gauge fixing condition must impose a constraint on the coordinates $A^i$—and perhaps their derivatives too—but no constraint should be placed on the conjugate momenta $\Pi^i$. The constraint $C_1$ is however, independent of the gauge fixing procedure. Thus, $C_1$ imposes a constraint on the momenta and $C_2$ imposes a constraint on the coordinates, thus it will generally be true that the constraint matrix will be of the form

$$C_{ab}(\vec{x},\vec{y}) = \{C_a(\vec{x}), C_b(\vec{y})\} \tag{44a}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab} \mathcal{D}\delta^3(\vec{x}-\vec{y}), \tag{44b}$$

where $\mathcal{D}$ is some operator. In what follows, we will assume that $\mathcal{D}$ is invertible. The inverse of the constraint matrix $C^{ab}$ will then be

$$C^{ab}(\vec{x},\vec{y}) = \delta^3(\vec{x}-\vec{y})\mathcal{D}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{ab}. \tag{45a}$$

Once gauge fixing has been imposed, we find that all constraints are second class, and we may strongly set these constraints equal to zero provided we use the Dirac bracket when computing time evolution of phase-space functions. Thus, the Hamiltonian we must use to compute any time derivatives (as was the case in eq. (28)) is of the form

$$\mathcal{H} = \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\Pi^i\Pi^i. \tag{46}$$

That is a gauge invariant statement. The following results will thus all be gauge invariant statements:

$$\{\Pi^i(\vec{x}), C_1(\vec{z}_1)\} = 0, \tag{47a}$$
$$\{\Pi^i, H\} = \nabla^2 A^i - \nabla^i \nabla \cdot \vec{A}, \tag{47b}$$
$$\{C_1(\vec{y}), H\} = 0, \tag{47c}$$
$$\{A^i(\vec{x}), C_1(\vec{y})\} = \nabla^i_y \delta^3(\vec{x} - \vec{y}), \tag{47d}$$
$$\{A^i(\vec{x}), H\} = \Pi^i(\vec{x}). \tag{47e}$$

Since $C^{ab}$ only contains off diagonal elements, one then finds:

$$\dot{\Pi}^i = \{\Pi^i(\vec{x}), H\}_D \tag{48a}$$
$$= \{\Pi^i(\vec{x}), H\} - \int d^3z_1 d^3z_2 \{\Pi^i(\vec{x}), C_a(\vec{z}_1)\} C^{ab}(\vec{z}_1, \vec{z}_2)\{C_b(\vec{z}_2), H\} \tag{48b}$$
$$= \{\Pi^i, H\} + 0 \tag{48c}$$
$$= \nabla^2 A^i - \nabla^i \nabla \cdot \vec{A}. \tag{48d}$$

Thus, we see $\dot{\Pi}^i$ is completely independent of gauge, which is expected as we showed that $\Pi^i$ was gauge invariant through eq. (22). The coordinates $A^i$ were however, shown not to be gauge invariant, and thus we should not expect $\dot{A}^i$ to be gauge invariant. Computing $\dot{A}^i$ using the Dirac bracket and under our general gauge choice, we find the following:

$$\dot{A}^i(\vec{x}) = \{A^i(\vec{x}), H\}_D \tag{49a}$$
$$= \{A^i(\vec{x}), H\} - \int d^3z_1 d^3z_2 \{A^i(\vec{x}), C_a(\vec{z}_1)\} C^{ab}(\vec{z}_1, \vec{z}_2)\{C_b(\vec{z}_2), H\} \tag{49b}$$
$$= \Pi^i(\vec{x}) - \int d^3z_1 d^3z_2 \{A^i(\vec{x}), C_1(\vec{z}_1)\} C^{12}(\vec{z}_1, \vec{z}_2) \overbrace{\{C_2(\vec{z}_2), H\}}^{\equiv f(C_2, H, \vec{z}_2)} \tag{49c}$$
$$= \Pi^i(\vec{x}) + \int d^3z_1 d^3z_2 \big(\nabla^i_{z_1} \delta^3(\vec{x} - \vec{z}_1)\big) \delta^3(\vec{z}_1 - \vec{z}_2) \mathcal{D}^{-1} f(C_2, H, \vec{z}_2) \tag{49d}$$
$$= \Pi^i(\vec{x}) + \int d^3z_1 \nabla^i_{z_1} \delta^3(\vec{x} - \vec{z}_1) \mathcal{D}^{-1} f(C_2, H, \vec{z}_1) \tag{49e}$$
$$\overset{\text{IBP}}{=} \Pi^i(\vec{x}) - \int d^3z_1 \delta^3(\vec{x} - \vec{z}_1) \nabla^i_{z_1} \mathcal{D}^{-1} f(C_2, H, \vec{z}_1) \tag{49f}$$
$$= \Pi^i(\vec{x}) - \nabla^i_x \mathcal{D}^{-1} f(C_2, H, \vec{x}) \tag{49g}$$

Since $A^i$ is not gauge invariant, it is not of physical interest. However, the curl of $A^i$ is of physical interest as this is the magnetic field. Computing the curl of $\dot{A}^i$, we find:

$$\epsilon^{kji} \partial^k \dot{A}^i = \epsilon^{kji} \partial^k \Pi^i + \overbrace{\epsilon^{kji} \partial^k \partial^i}^{\text{0 by anti-symmetry}} \mathcal{D}^{-1} f(C_2, H, \vec{x}) \tag{50a}$$
$$\partial_t \epsilon^{kji} \partial^k A^i = \epsilon^{kji} \partial^k E^i \tag{50b}$$
$$\partial_t (\nabla \times \vec{A}) = (\nabla \times \vec{E})^k \tag{50c}$$
$$\implies \partial_t \vec{B} = \nabla \times \vec{E} \tag{50d}$$

Notice that all the gauge dependency was contained in the term $\nabla_x^i \mathcal{D}^{-1} f(C_2, H, \vec{x})$, which vanishes once one takes the curl.

Thus, we see that one may derive Maxwell's equations independently of the gauge we choose. Whether the gauge choice is non-integrable non-holonomic is ultimately inconsequential, all that is required is that the operator $\mathcal{D}$ is invertible.

## 2.3 'ʼt Hooft-Veltman Gauge

Consider the 't Hooft-Veltman Gauge, which (in the Hamilton formalism) is defined by the following constraint on the fields

$$C_2 \equiv \nabla \cdot \vec{A} + \gamma \vec{A} \cdot \vec{A} = 0. \tag{51}$$

Computing the Poisson bracket of the constraints in the 't Hooft-Veltman gauge, one finds:

$$\begin{aligned}
\{C_1(\vec{x}), C_2(\vec{y})\} &= \{\nabla_x^j \Pi^j(\vec{x}), \nabla_y^i A^i(\vec{y}) + \gamma A^i(\vec{y}) A^i(\vec{y})\} && \text{(52a)}\\
&= \nabla_x^2 \delta^3(\vec{x} - \vec{y}) + \{\nabla_x^j \Pi^j(\vec{x}), \gamma A^i(\vec{y}) A^i(\vec{y})\} && \text{(52b)}\\
&= (\nabla_x^2 + 2\gamma A^i(\vec{y}) \nabla_x^i) \delta^3(\vec{x} - \vec{y}) && \text{(52c)}\\
&= \mathcal{D}_{tHV} \delta^3(\vec{x} - \vec{y}), && \text{(52d)}
\end{aligned}$$

where we have defined the operator

$$\mathcal{D}_{tHV} \equiv \nabla_x^2 + 2\gamma A^i(\vec{y}) \nabla_x^i. \tag{53}$$

The constraint matrix is then

$$\begin{aligned}
C_{ab}(\vec{x}, \vec{y}) &= \{C_a(\vec{x}), C_b(\vec{y})\} && \text{(54a)}\\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab} \mathcal{D}_{tHV} \delta^3(\vec{x} - \vec{y}), && \text{(54b)}
\end{aligned}$$

which has the inverse

$$C^{ab}(\vec{x}, \vec{y}) = \delta^3(\vec{x} - \vec{y}) \mathcal{D}_{tHV}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{ab}. \tag{55a}$$

The $\mathcal{D}_{tHV}^{-1}$ operator is formally given by

$$\mathcal{D}_{tHV}^{-1} = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\vec{k}(\vec{x}-\vec{y})}}{\vec{k}^2 + 2i\gamma \vec{k} \cdot \vec{A}(\vec{y})}. \tag{56}$$

Using the general discussion we presented in section 2.2, we will find the correct equaitions of motion for the electric and magnetic fields in the 't Hooft-Veltman gauge.

# References

[1] P.A.M. Dirac. *Lectures on Quantum Mechanics*. Dover Books on Physics. Dover Publications, 2013.

[2] Steven Weinberg. *The Quantum Theory of Fields*. Cambridge University Press, 1995.