R_ξ Gauge Fixing

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Abstract

Here, I would like to outline the R_ξ gauge fixing procedure for the Maxwell Lagrangian of electrodynamics. The R_ξ gauge fixing procedure is a powerful method for imposing gauge fixing constraints at the level of the Lagrangian, and it is widely used in quantum field theory and gauge theories. In what follows, I will provide a step-by-step outline of the R_ξ gauge fixing procedure, starting from the Maxwell Lagrangian and leading to the master equation for gauge fixing.

To begin, consider the Maxwell Lagrangian of electrodynamics

$$\mathscr{L}_{EM} = -\frac{1}{4}F^2 - j_\mu A^\mu, \tag{1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength tensor and j_μ is a conserved current. The Maxwell Lagrangian is gauge invariant, and one may freely impose a gauge fixing constraint on the A^μ fields to remove redundant degrees of freedom. Gauge fixing from the level of the Lagrangian can be achieved by making use of the R_ξ gauge fixing procedure.

Suppose we wish to impose the gauge fixing condition $C = C[A^\mu, \partial_\nu A^\mu]$ on the A^μ fields. The R_ξ gauge fixing procedure instructs one to consider the following Lagrangian

$$\mathscr{L} = \mathscr{L}_{EM} + BC + f[B,\xi], \tag{2}$$

where $B = B(x)$ are the Nakanishi-Lautrup fields—which act similarly to Lagrange multipliers—and f is an arbitrary functional of the B fields and a gauge fixing parameter ξ. Now, upon varying the action corresponding to the Lagrangian in eq. (2), one finds:

$$\delta S = \int d^4x\, \delta\mathscr{L} \tag{3a}$$

$$= \int d^4x\, \delta\mathscr{L}_{EM} + B\delta C + C\delta B + \delta f \tag{3b}$$

$$= \int d^4x\, \left(\frac{\partial \mathscr{L}_{EM}}{\partial A^\mu} + B\frac{\partial C}{\partial A^\mu}\right)\delta A^\mu + \left(\frac{\partial \mathscr{L}_{EM}}{\partial(\partial_\nu A^\mu)} + B\frac{\partial C}{(\partial_\nu A^\mu)}\right)\delta(\partial_\nu A^\mu) + \left(C + \frac{\partial f}{\partial B}\right)\delta B \tag{3c}$$

$$= \int d^4x\, \left(\frac{\partial \mathscr{L}_{EM}}{\partial A^\mu} + B\frac{\partial C}{\partial A^\mu} - \partial_\nu\left[\frac{\partial \mathscr{L}_{EM}}{\partial(\partial_\nu A^\mu)} + B\frac{\partial C}{(\partial_\nu A^\mu)}\right]\right)\delta A^\mu + \left(C + \frac{\partial f}{\partial B}\right)\delta B. \tag{3d}$$

Note that in passing from eq. (3c) to eq. (3d) we assumed the *transposition rule* and then integrated by parts. Varying the action with respect to the Nakanishi-Lautrup fields, one finds

$$C = -\frac{\partial f}{\partial B} = \frac{B}{\xi}, \tag{4}$$

where the second equality follows if one chooses the arbitrary functional f as

$$f[B, \xi] = -\frac{B^2}{2\xi}. \tag{5}$$

Upon varying the action with respect to the A^μ fields and making use of eq. (4), one finds:

$$0 = \frac{\partial \mathscr{L}_{EM}}{\partial A^\mu} + \xi C \frac{\partial C}{\partial A^\mu} - \partial_\nu \left[\frac{\partial \mathscr{L}_{EM}}{\partial(\partial_\nu A^\mu)} + \xi C \frac{\partial C}{(\partial_\nu A^\mu)} \right] \tag{6}$$

$$\implies j^\mu = \Box A^\mu - \partial^\mu \partial_\nu A^\nu + \xi \left(C \frac{\partial C}{\partial A_\mu} - \partial_\nu \left[C \frac{\partial C}{(\partial_\nu A_\mu)} \right] \right). \tag{7}$$