Renormalization of QED

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Abstract

These notes were made in an attempt to teach myself how to renormalize Quantum Electrodynamics (QED). In no way are these notes a comprehensive review of the subject, and no new results are presented here. We begin by calculating all divergent QED diagrams to order $\mathcal{O}(\alpha)$ *via* dimensional regularization, and then proceed to renormalize the theory by introducing counterterms. We then demonstrate how the corrections we have calculated lead to the running of the QED coupling constant. The contents presented here are based off the books of Ryder [1], Peskin and Schroeder [2], Sidney Coleman [4], and Gelis [3].

Contents

1 Bare and Renormalized Lagrangian

Consider the bare (gauge fixed) Lagrangian of QED

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{B,\mu\nu}F_B^{\mu\nu} + \bar{\psi}_B(i\partial\!\!\!/ - m_B)\psi_B - e_B\bar{\psi}_B A\!\!\!/_B\psi_B - \frac{1}{2\xi_B}(\partial_\mu A_B^\mu)^2. \tag{1.1}$$

In eq. (1.1), the subscript B denotes a bare quantity, which is not a directly measurable quantity. The field strength tensor is defined as $F_{B,\mu\nu} = \partial_\mu A_{B,\nu} - \partial_\nu A_{B,\mu}$. In this discussion, we work in a spacetime dimension of $D = 4-\epsilon$, where ϵ is a small real number. Note that the bare coupling, e_B, has mass dimension $\epsilon/2$. The bare quantities are related to the renormalized quantities *via* the following relations:

$$A_B^\mu = \sqrt{Z_3}A^\mu, \quad \psi_B = \sqrt{Z_2}\psi, \quad m_B = m + \delta_m, \quad e_B = Z_1 Z_2^{-1} Z_3^{-1/2}\mu^{\epsilon/2}e, \quad \xi_B = Z_3\xi, \tag{1.2}$$

where A^μ and ψ are the renormalized photon and electron fields, respectively; m is the renormalized electron mass, and e is the renormalized coupling constant. The Z_i terms are the renormalization constants, which are defined such that $Z_i \equiv 1 + \delta_i$, where δ_i is a counterterm that is chosen to cancel the divergences in the theory (more on this later). The term μ is an arbitrary mass scale that is introduced to keep the renormalized coupling constant e dimensionless in $D = 4 - \epsilon$ dimensions. Lastly, ξ_B is the bare gauge fixing parameter, and ξ is the renormalized gauge fixing parameter.

The Lagrangian in eq. (1.1) can be rewritten in terms of the renormalized quantities as

$$\begin{aligned}\mathscr{L}_{\text{QED}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \mu^{\epsilon/2}e\bar{\psi}A\!\!\!/\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 &(1.3\text{a})\\ &-\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \delta_2\bar{\psi}i\partial\!\!\!/\psi - (m\delta_2 + \delta_m)\bar{\psi}\psi - \delta_1\mu^{\epsilon/2}e\bar{\psi}A\!\!\!/\psi \\ = &\ \mathscr{L}_{\text{Renormalized}} + \mathscr{L}_{\text{Counterterms}}, &(1.3\text{b})\end{aligned}$$

with

$$\mathscr{L}_{\text{Renormalized}} \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \mu^{\epsilon/2}e\bar{\psi}A\!\!\!/\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{1.4a}$$

$$\mathscr{L}_{\text{Counterterms}} \equiv -\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \delta_2\bar{\psi}i\partial\!\!\!/\psi - (m\delta_2 + \delta_m)\bar{\psi}\psi - \delta_1\mu^{\epsilon/2}e\bar{\psi}A\!\!\!/\psi \tag{1.4b}$$

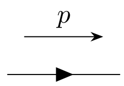
Note that the counterterms in $\mathscr{L}_{\text{Counterterms}}$ are (as we will show later) all $\mathcal{O}(e^2)$, and so eq. (1.3a) is equivalent to the bare Lagrangian in eq. (1.1) up to $\mathcal{O}(e^4)$.

2 Feynman Rules

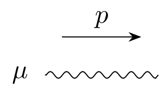
In what follows, we will set $\xi = 1$, thereby placing us in the Feynman gauge. With this choice, the renormalized Lagrangian becomes

$$\mathscr{L}_{\text{Renormalized}} = \frac{1}{2}g_{\mu\nu}A^\mu\Box A^\nu + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \mu^{\epsilon/2}e\bar{\psi}A\!\!\!/\psi \tag{2.1}$$

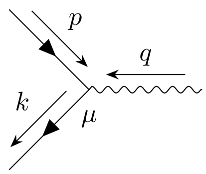
From eq. (2.1), we can read off the corresponding Feynman rules as:



$$= S_F^0(p) = \frac{i(\not{p}+m)}{p^2-m^2+i0^+}$$



$$= D_F^{0,\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2+i0^+}$$

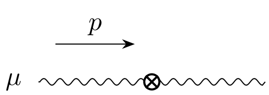


$$= -i\mu^{\epsilon/2}e\gamma^\mu$$

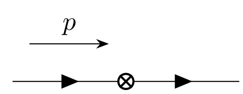
Next, we rewrite the counterterm Lagrangian in eq. (1.3a) as

$$\mathscr{L}_{\text{Counterterms}} = \frac{1}{2}\delta_3 A^\nu\left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)A^\mu + \delta_2\bar{\psi}i\not{\partial}\psi - (m\delta_2+\delta_m)\bar{\psi}\psi - \delta_1\mu^{\epsilon/2}e\bar{\psi}\not{A}\psi + \text{Total Derivatives.} \quad (2.2)$$

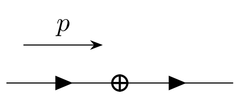
Ignoring the total derivative terms which do not contribute to the action, we can read off the corresponding Feynman rules for the counterterm Lagrangian as:



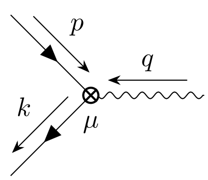
$$= -i\delta_3\left(p^2g^{\mu\nu} - p^\mu p^\nu\right)$$



$$= i\delta_2\not{p}$$



$$= -i(m\delta_2+\delta_m)$$

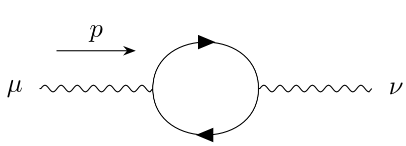


$$= -i\mu^{\epsilon/2}e\delta_1\gamma^\mu$$

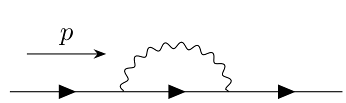
To get these rules, one should interpret each counterterm as an interaction term in $\mathscr{L}_{\text{QED}}$.

3 Divergent One-Loop Diagrams

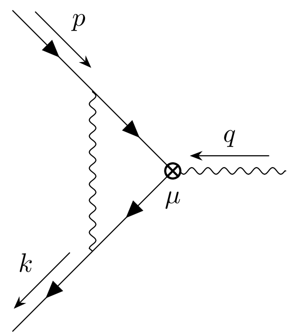
In QED, to one loop (equivalently $\mathcal{O}(\alpha) = \mathcal{O}(e^2)$), the only divergent (and hence non-zero) diagrams are the electron self-energy, the photon self-energy, and the vertex correction (shown below).



$$\equiv -i\Pi^{\mu\nu}(p)$$



$$\equiv i\Sigma(p)$$



These divergent diagrams are calculated in appendix A using dimensional regularization. Here we state the results of these calculations:

$$-i\Pi^{\mu\nu}(p) = -i\frac{e^2}{2\pi^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left[\frac{1}{3\epsilon} - \frac{\gamma_E}{6} - \int_0^1 dx\, x(1-x)\log\left(\frac{-m^2 + p^2x(1-x)}{4\pi\mu^2}\right)\right] + \mathcal{O}(\epsilon) \tag{3.1a}$$

$$= -i\frac{e^2}{6\pi^2\epsilon}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right) - i\Pi^{\mu\nu}_{\text{finite}}(p) \tag{3.1b}$$

$$i\Sigma(p) = \frac{ie^2}{8\pi^2\epsilon}(\not{p} - 4m) + \frac{ie^2}{16\pi^2}\Bigg[2m(1+2\gamma_E) - \not{p}(1+\gamma_E) \tag{3.1c}$$

$$- 2\int_0^1 dx\,\left(\not{p}(1-x) - 2m\right)\log\left(\frac{-m^2x + p^2x(1-x)}{4\pi\mu^2}\right)\Bigg] + \mathcal{O}(\epsilon)$$

$$= \frac{ie^2}{8\pi^2\epsilon}(\not{p} - 4m) + i\Sigma_{\text{finite}}(p) \tag{3.1d}$$

$$-i\mu^{\epsilon/2}e\Lambda^\mu(p,k,q) = -i\mu^{\epsilon/2}e\Bigg\{\frac{e^2}{8\pi^2\epsilon} - \frac{e^2}{16\pi^2}\Bigg[\gamma_E + 2 \tag{3.1e}$$

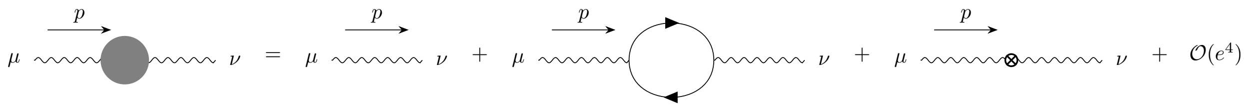
$$+ 2\int_0^1 dx\int_0^{1-x} dy\;\log\left(\frac{-m^2(x+y) + p^2x(1-x) + k^2y(1-y) - 2p\cdot kxy}{4\pi\mu^2}\right)\Bigg]$$

$$+ \int_0^1 dx\int_0^{1-x} dy\;\frac{\gamma^\nu(\not{p}(1-x) - \not{k}y + m)\gamma^\mu(\not{k}(1-y) - \not{p}x + m)\gamma_\nu}{m^2(x+y) - p^2x(1-x) - k^2y(1-y) + 2p\cdot kxy} + \mathcal{O}(\epsilon)\Bigg\}.$$

$$= -i\mu^{\epsilon/2}e\left(\frac{e^2}{8\pi^2\epsilon}\right)\gamma^\mu - i\mu^{\epsilon/2}e\Lambda^\mu_{\text{finite}}(p,k,q) \tag{3.1f}$$

In each of the above, we have separated the divergent (in the limit $\epsilon \to 0$) and finite parts of the respective diagrams.

Next, we explain how to choose the counterterms δ_1, δ_2, δ_3, and δ_m such that the divergences in the theory are canceled, which will thereby leave the respective two and three point functions finite. We begin by considering the *fully dressed* photon propagator, $D_F^{\mu\nu}$, which to order $\mathcal{O}(e^2)$ has the following diagrammatic representation:



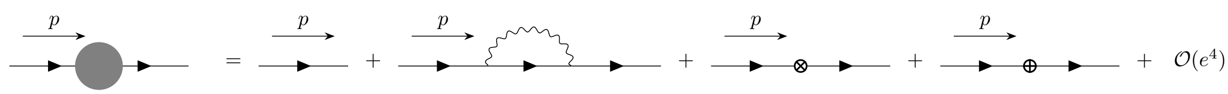
Then, by making use of the Feynman rules above along with eq. (3.1b), we can write the fully dressed photon propagator as

$$D_F^{\mu\nu}(p) = -i\frac{g^{\mu\nu}}{p^2+i0^+} - i\frac{e^2}{6\pi^2\epsilon}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right) - i\Pi^{\mu\nu}_{\text{finite}}(p) - i\delta_3\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right) + \mathcal{O}(e^4). \tag{3.2}$$

From eq. (3.2), to cancel the divergence in the fully dressed photon propagator, we set the counterterm δ_3 to be

$$\delta_3 = -\frac{e^2}{6\pi^2\epsilon}. \tag{3.3}$$

We now consider the fully dressed electron propagator, $S_F(p)$, which to order $\mathcal{O}(e^2)$ has the following diagrammatic representation



We may then again make use of the Feynman rules along with eq. (3.1d) to write the fully dressed electron propagator as

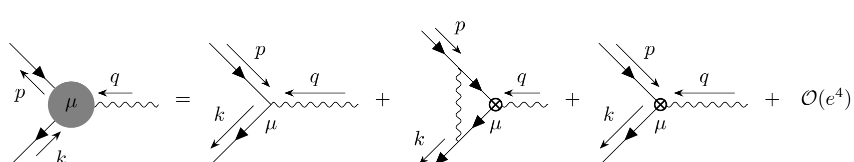
$$S_F(p) = i\frac{\not{p}+m}{p^2-m^2+i0^+} + \frac{ie^2}{8\pi^2\epsilon}(\not{p}-4m) + i\Sigma_{\text{finite}}(p) + i\delta_2\not{p} - i(m\delta_2+\delta_m) + \mathcal{O}(e^4). \tag{3.4}$$

It is then obvious to see that we should choose the counterterms δ_2 and δ_m to be:

$$\delta_2 = -\frac{e^2}{8\pi^2\epsilon}, \tag{3.5a}$$

$$\delta_m = -\frac{3me^2}{8\pi^2\epsilon} \tag{3.5b}$$

Repeating this process for the fully dressed vertex function, Γ^μ, which to order $\mathcal{O}(e^2)$ has the following diagrammatic representation



Then, by making use of the Feynman rules along with eq. (3.1f), we can write the fully dressed vertex function as

$$\Gamma^\mu(p,k,q) = -i\mu^{\epsilon/2}e\gamma^\mu - i\mu^{\epsilon/2}e\left(\frac{e^2}{8\pi^2\epsilon}\right)\gamma^\mu - i\mu^{\epsilon/2}e\Lambda^\mu_{\text{finite}}(p,k,q) - i\mu^{\epsilon/2}e\delta_1\gamma^\mu + \mathcal{O}(e^4). \tag{3.6}$$

We see that the divergence in the fully dressed vertex function is canceled by setting the counterterm δ_1 to be

$$\delta_1 = -\frac{e^2}{8\pi^2\epsilon}. \tag{3.7}$$

Notice that $Z_1 \equiv 1 + \delta_1 = 1 + \delta_2 \equiv Z_2$, this relation is in fact a consequence of the Ward identity.

4 Running of the QED Coupling

The introduction of the renormalization scale μ in the renormalized Lagrangian in eq. (2.1) may naively feel as a trick to account for the arbitrary integration dimensions used in dimensional regularization. However, the introduction of μ leads to direct physical consequences that are confirmed by experiment. In this section, we will show how the introduction of μ leads to the running of the QED coupling e with energy scale.

The quantities of the bare Lagrangian in eq. (1.1) are all independent of the renormalization scale μ. In particular, the bare coupling e_B is independent of μ. From eq. (1.2), we have

$$e_B = Z_3^{-1/2}\mu^{\epsilon/2}e = \frac{1}{\sqrt{1-\frac{e^2}{6\pi^2\epsilon}}}\mu^{\epsilon/2}e, \tag{4.1}$$

where we have made use of the fact that $Z_1 = Z_2$, and we have used eq. (3.3) to obtain the expression for Z_3 explicitly. For e_B to be independent of μ, it is clear from eq. (4.1) that the renormalized coupling e must depend on μ. To investigate this dependence, we differentiate both sides of eq. (4.1) with respect to μ:

$$0 = \frac{\partial}{\partial\mu}\left[\mu^{\epsilon/2}e\left(1+\frac{e^2}{12\pi^2\epsilon}\right)\right] + \mathcal{O}(e^4) \tag{4.2a}$$

$$\implies \mu\frac{\partial e}{\partial\mu} = \frac{e^3}{12\pi^2} - \frac{\epsilon e}{2} + \mathcal{O}(e^4) \tag{4.2b}$$

$$\lim_{\epsilon\to 0}\mu\frac{\partial e}{\partial\mu} = \frac{e^3}{12\pi^2} \tag{4.2c}$$

The ODE in eq. (4.2c) is solved by

$$e^2(\mu) = \frac{e^2(\mu_0)}{1-\frac{e^2(\mu_0)}{6\pi^2}\log\left(\frac{\mu}{\mu_0}\right)} \tag{4.3}$$

where μ_0 is some reference scale at which the coupling is measured to be $e(\mu_0)$. The QED β function (defined as $\beta \equiv \mu\partial_\mu e$) is positive, which means that the QED coupling e increases with energy scale μ; this is also clearly apparent from eq. (4.3). At higher energy scales, the QED coupling e becomes larger, which means that the perturbation theory becomes less reliable. This is the exact opposite behavior of the QCD coupling, which has a negative β function and becomes weaker at higher energy scales.

5 Summary

In these notes, we have renormalized QED to one loop by calculating the theory's three divergent diagrams—the photon self-energy, electron self-energy, and vertex correction—*via* dimensional regularization, and by the fixing of the counterterms δ_1, δ_2, δ_3, and δ_m to render the dressed propagators and vertex finite. In doing so, we found $Z_1 = Z_2$, a direct consequence of the Ward identity. We showed that the renormalized coupling, e, must depend on the renormalization scale, μ, and derived the one-loop QED β function, $\beta(e) = e^3/(12\pi^2)$. Since $\beta > 0$, the QED coupling grows with energy scale.

References

[1] L. H. Ryder. *Quantum Field Theory*. Cambridge University Press, 6 1996.

[2] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[3] François Gelis. *Quantum Field Theory*. Cambridge University Press, 7 2019.

[4] Sidney Coleman. *Lectures of Sidney Coleman on Quantum Field Theory*. WSP, Hackensack, 12 2018.

A Calculation of One-Loop Feynman Diagrams

In this section, we explicitly calculate the divergent one-loop diagrams whose results presented in eq. (3.1). To perform these calculations, we will make use of the following identities:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2} \tag{A.1a}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[A(1-x-y) + Bx + Cy]^3} \tag{A.1b}$$

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(k_E^2 + \Delta)^n} = \frac{\Delta^{\frac{D}{2}-n}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} \tag{A.1c}$$

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{(k_E)^2}{(k_E^2 + \Delta)^n} = \frac{1}{2} \frac{\Delta^{\frac{D}{2}+1-n}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(n - 1 - \frac{D}{2}\right)}{\Gamma(n)} \tag{A.1d}$$

$$\mathrm{tr}(\gamma^{\mu_1} \dots \gamma^{\mu_n}) = 0, \quad n = \text{ odd} \tag{A.1e}$$

$$\mathrm{tr}[\mathbb{1}] = f(D) \tag{A.1f}$$

$$\mathrm{tr}[\gamma^\mu \gamma^\nu] = f(D)\, g^{\mu\nu} \tag{A.1g}$$

$$\mathrm{tr}[\gamma^\mu \not{a}\, \gamma^\nu \not{b}] = f(D)\, (a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu}(a \cdot b)) \tag{A.1h}$$

$$\gamma^\mu \gamma_\mu = D \tag{A.1i}$$

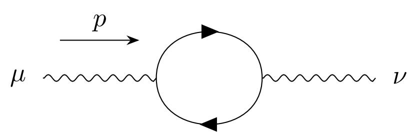
$$\gamma^\mu \not{a}\, \gamma_\mu = (2-D)\not{a} \tag{A.1j}$$

$$\gamma^\nu \gamma^\sigma \gamma^\mu \gamma_\sigma \gamma_\nu = (2-D)^2 \gamma^\mu \tag{A.1k}$$

In the above, $f(D)$ is a smooth function of D such that $f(4) = 4$.

A.1 Photon Self-Energy

The photon self-energy is given by the following diagram



$$\equiv -i\Pi^{\mu\nu}(p).$$

From the Feynman rules, we can write down the Feynman integral for the photon self-energy as:

$$-i\Pi^{\mu\nu}(p) = -\left(-i\mu^{\epsilon/2}e\right)^2 \int \frac{d^Dq}{(2\pi)^D}\,\mathrm{tr}\left[\gamma^\mu \frac{i(\not{q}+m)}{q^2-m^2+i0^+}\gamma^\nu \frac{i(\not{q}-\not{p}+m)}{(q-p)^2-m^2+i0^+}\right] \tag{A.2a}$$

$$= -\mu^\epsilon e^2 \int \frac{d^Dq}{(2\pi)^D}\,\mathrm{tr}\left[\gamma^\mu(\not{q}+m)\gamma^\nu(\not{q}-\not{p}+m)\right]\frac{1}{\left(q^2-m^2+i0^+\right)\left((q-p)^2-m^2+i0^+\right)} \tag{A.2b}$$

$$= -\mu^\epsilon e^2 \int \frac{d^Dq}{(2\pi)^D}\int_0^1 dx \frac{\mathrm{tr}\left[\gamma^\mu(\not{q}+m)\gamma^\nu(\not{q}-\not{p}+m)\right]}{\left[x\left(q^2-m^2\right)+(1-x)\left((q-p)^2-m^2\right)+i0^+\right]^2} \tag{A.2c}$$

The overall minus sign in eq. (A.2a) comes from the fermion loop; in eq. (A.2c), we have used the Feynman parametrization identity in eq. (A.1a). Next, we complete the square in the denominator of eq. (A.2c), which gives:

$$-i\Pi^{\mu\nu}(p) = -\mu^\epsilon e^2 \int \frac{d^Dq}{(2\pi)^D}\int_0^1 dx \frac{\mathrm{tr}\left[\gamma^\mu(\not{q}+m)\gamma^\nu(\not{q}-\not{p}+m)\right]}{\left[(q-px)^2-(m^2+p^2x(x-1))+i0^+\right]^2} \tag{A.3a}$$

$$= -\mu^\epsilon e^2 \int \frac{d^Dq}{(2\pi)^D}\int_0^1 dx \frac{\mathrm{tr}\left[\gamma^\mu(\not{q}+\not{p}x+m)\gamma^\nu(\not{q}+\not{p}(x-1)+m)\right]}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]^2} \tag{A.3b}$$

To proceed from this point, we need to evaluate the trace in the numerator of the integrand. We have:

$$\begin{aligned}\mathrm{tr}\left[\gamma^\mu(\not{q}+x\not{p}+m)\gamma^\nu(\not{q}+(x-1)\not{p}+m)\right] &= \mathrm{tr}[\gamma^\mu\not{q}\gamma^\nu\not{q}] + (x-1)\,\mathrm{tr}[\gamma^\mu\not{q}\gamma^\nu\not{p}] + m\,\mathrm{tr}[\gamma^\mu\not{q}\gamma^\nu] \qquad \text{(A.4a)}\\ &+ x\,\mathrm{tr}[\gamma^\mu\not{p}\gamma^\nu\not{q}] + x(x-1)\,\mathrm{tr}[\gamma^\mu\not{p}\gamma^\nu\not{p}] + xm\,\mathrm{tr}[\gamma^\mu\not{p}\gamma^\nu]\\ &+ m\,\mathrm{tr}[\gamma^\mu\gamma^\nu\not{q}] + m(x-1)\,\mathrm{tr}[\gamma^\mu\gamma^\nu\not{p}] + m^2\,\mathrm{tr}[\gamma^\mu\gamma^\nu]\\ &= \mathrm{tr}[\gamma^\mu\not{q}\gamma^\nu\not{q}] + x(x-1)\,\mathrm{tr}[\gamma^\mu\not{p}\gamma^\nu\not{p}] + m^2\,\mathrm{tr}[\gamma^\mu\gamma^\nu] \qquad \text{(A.4b)}\\ &+ (x-1)\,\mathrm{tr}[\gamma^\mu\not{q}\gamma^\nu\not{p}] + x\,\mathrm{tr}[\gamma^\mu\not{p}\gamma^\nu\not{q}]\end{aligned}$$

To get to eq. (A.4b), we have used the identity in eq. (A.1e). Notice that the last two terms in eq. (A.4b) are odd in q, and therefore vanish upon integration over the symmetric q domain; thus, in what follows, we will ignore these terms. We then make use of the trace identities in eq. (A.1) to evaluate the remaining traces, in doing so, we find:

$$\mathrm{tr}\left[\gamma^\mu(\not{q}+x\not{p}+m)\gamma^\nu(\not{q}+(x-1)\not{p}+m)\right] = f(D)\Big[\left(2q^\mu q^\nu - g^{\mu\nu}q^2\right) + x(x-1)\left(2p^\mu p^\nu - g^{\mu\nu}p^2\right) + m^2 g^{\mu\nu}\Big] \tag{A.5a}$$

$$\begin{aligned}&= f(D)\Big[2q^\mu q^\nu + 2x(x-1)(p^\mu p^\nu - g^{\mu\nu}p^2)\\ &\quad - g^{\mu\nu}\left(q^2 - \left(m^2 + x(x-1)p^2\right)\right)\Big].\end{aligned} \tag{A.5b}$$

Returning to the expression for the photon self-energy, we have

$$-i\Pi^{\mu\nu}(p) = -\mu^{\epsilon}e^2 f(D)\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dx\Bigg\{\frac{2q^\mu q^\nu}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]^2} + \frac{2x(x-1)(p^\mu p^\nu - g^{\mu\nu}p^2)}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]^2} - \frac{g^{\mu\nu}}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]}\Bigg\}. \tag{A.6}$$

The first and third terms in the integrand above cancel each other upon integration over q.To see this, consider the first term in the integrand:

$$\int\frac{d^Dq}{(2\pi)^D}\frac{2q^\mu q^\nu}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]^2} = 2g^{\mu\nu}\int\frac{d^Dq}{(2\pi)^D}\frac{q^2}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]^2} \tag{A.7a}$$

$$= 2g^{\mu\nu}\int\frac{d^Dq_E}{(2\pi)^D}\frac{-iq_E^2}{\left[q_E^2+(m^2+p^2x(x-1))\right]^2} \tag{A.7b}$$

$$= -2ig^{\mu\nu}\left[\frac{\left(m^2+p^2x(x-1)\right)^{D/2-1}}{2(4\pi)^{D/2}}\frac{\Gamma(1-D/2)}{\Gamma(2)}\right] \tag{A.7c}$$

$$= -ig^{\mu\nu}\left[\frac{\left(m^2+p^2x(x-1)\right)^{D/2-1}}{(4\pi)^{D/2}}\Gamma(1-D/2).\right] \tag{A.7d}$$

In eq. (A.7b), we have performed a Wick rotation to Euclidean space, and in eq. (A.7c), we have used the identity in eq. (A.1d). Next, consider the third term in the integrand:

$$\int\frac{d^Dq}{(2\pi)^D}\frac{g^{\mu\nu}}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]} = g^{\mu\nu}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]} \tag{A.8a}$$

$$= g^{\mu\nu}\int\frac{d^Dq_E}{(2\pi)^D}\frac{-i}{\left[q_E^2+(m^2+p^2x(x-1))\right]} \tag{A.8b}$$

$$= -ig^{\mu\nu}\left[\frac{\left(m^2+p^2x(x-1)\right)^{D/2-1}}{(4\pi)^{D/2}}\Gamma(1-D/2).\right] \tag{A.8c}$$

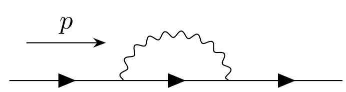
In eq. (A.8b), we have again performed a Wick rotation to Euclidean space, and in eq. (A.8c), we have used the identity in eq. (A.1c). Thus, we have confirmed the claim that the first and third terms in the integrand cancel each other upon integration over q. We are then left with:

$$
\begin{aligned}
-i\Pi^{\mu\nu}(p) &= -\mu^{\epsilon}e^2 f(D)\int \frac{d^Dq}{(2\pi)^D}\int_0^1 dx \frac{2x(x-1)(p^\mu p^\nu - g^{\mu\nu}p^2)}{\left[q^2-(m^2+p^2x(x-1))+i0^+\right]^2} && \text{(A.9a)}\\
&= -\mu^{\epsilon}e^2 f(D)\int_0^1 dx\int \frac{d^Dq_E}{(2\pi)^D}\frac{2ix(x-1)(p^\mu p^\nu - g^{\mu\nu}p^2)}{\left[q_E^2+(m^2+p^2x(x-1))\right]^2} && \text{(A.9b)}\\
&= -\mu^{\epsilon}e^2 f(D)\int_0^1 dx 2ix(x-1)(p^\mu p^\nu - g^{\mu\nu}p^2)\frac{\left(m^2+p^2x(x-1)\right)^{D/2-2}\Gamma(2-D/2)}{(4\pi)^{D/2}\Gamma(2)} && \text{(A.9c)}\\
&= \frac{-2i\mu^{\epsilon}e^2 f(4-\epsilon)(p^\mu p^\nu - g^{\mu\nu}p^2)\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}}\int_0^1 dx x(x-1)\left(m^2+p^2x(x-1)\right)^{-\epsilon/2} && \text{(A.9d)}\\
&= \frac{-2ie^2 f(4-\epsilon)(p^\mu p^\nu - g^{\mu\nu}p^2)}{(4\pi)^2}\int_0^1 dx x(x-1)\Gamma(\epsilon/2)\left(\frac{-m^2+p^2x(1-x)}{4\pi\mu^2}\right)^{-\epsilon/2}(-1)^{-\epsilon/2} && \text{(A.9e)}\\
&= \frac{-2ie^2 f(4-\epsilon)(p^\mu p^\nu - g^{\mu\nu}p^2)}{(4\pi)^2}\int_0^1 dx x(x-1) && \text{(A.9f)}\\
&\times\left[\frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right]\left[1-\frac{\epsilon}{2}\log\left(\frac{-m^2+p^2x(1-x)}{4\pi\mu^2}\right)+\mathcal{O}(\epsilon^2)\right]\left[1+\mathcal{O}(\epsilon^2)\right]\\
&= \frac{-ie^2(g^{\mu\nu}p^2-p^\mu p^\nu)}{2\pi^2}\left[\frac{1}{3\epsilon}-\frac{\gamma_E}{6}-\int_0^1 dx\, x(1-x)\log\left(\frac{-m^2+p^2x(1-x)}{4\pi\mu^2}\right)\right]+\mathcal{O}(\epsilon). && \text{(A.9g)}
\end{aligned}
$$

We thus see that eq. (A.9g) agrees with the result in eq. (3.1b).

A.2 Electron Self Energy

The electron self-energy is given by the following diagram



$\equiv i\Sigma(p)$

Using the Feynman rules, we can write down the Feynman integral for the electron self-energy as:

$$
\begin{aligned}
i\Sigma(p) &= \left(-i\mu^{\epsilon/2}e\right)^2\int\frac{d^Dq}{(2\pi)^D}\gamma^\mu\frac{i(\not{p}-\not{q}+m)}{(p-q)^2-m^2+i0^+}\gamma^\nu\frac{-ig_{\mu\nu}}{q^2+i0^+} && \text{(A.10a)}\\
&= -\mu^\epsilon e^2\int\frac{d^Dq}{(2\pi)^D}\frac{\gamma^\mu(\not{p}-\not{q}+m)\gamma_\mu}{\left((p-q)^2-m^2+i0^+\right)(q^2+i0^+)} && \text{(A.10b)}\\
&= -\mu^\epsilon e^2\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dx\frac{\gamma^\mu(\not{p}-\not{q}+m)\gamma_\mu}{\left[x\left((p-q)^2-m^2\right)+(1-x)q^2+i0^+\right]^2} && \text{(A.10c)}
\end{aligned}
$$

In eq. (A.10c), we have used the Feynman parametrization identity in eq. (A.1a). Next, we complete the square in the denominator of eq. (A.10c), which gives:

$$
\begin{aligned}
i\Sigma(p) &= -\mu^\epsilon e^2\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dx\frac{\gamma^\mu(\not{p}-\not{q}+m)\gamma_\mu}{\left[(q-px)^2-(m^2x-p^2x(1-x))+i0^+\right]^2} && \text{(A.11a)}\\
&= -\mu^\epsilon e^2\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dx\frac{\gamma^\mu(\not{p}(1-x)-\not{q}+m)\gamma_\mu}{\left[q^2-(m^2x-p^2x(1-x))+i0^+\right]^2} && \text{(A.11b)}
\end{aligned}
$$

The linear term in q in the numerator of the integrand vanishes upon integration over the symmetric q domain. We then perform a Wick rotation to Euclidean space, which gives:

$$i\Sigma(p) = -i\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^D q_E}{(2\pi)^D} \frac{\gamma^\mu(\not{p}(1-x)+m)\gamma_\mu}{\left[q_E^2 + (m^2 x - p^2 x(1-x))\right]^2} \tag{A.12a}$$

$$= -i\mu^\epsilon e^2 \int_0^1 dx \frac{\gamma^\mu(\not{p}(1-x)+m)\gamma_\mu}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Gamma(2)} \left(m^2 x - p^2 x(1-x)\right)^{D/2-2} \tag{A.12b}$$

$$= \frac{-i\mu^\epsilon e^2}{(4\pi)^{2-\epsilon/2}} \int_0^1 dx \gamma^\mu(\not{p}(1-x)+m)\gamma_\mu \Gamma(\epsilon/2) \left(m^2 x - p^2 x(1-x)\right)^{-\epsilon/2} \tag{A.12c}$$

In eq. (A.12b), we have used the identity in eq. (A.1c), and in eq. (A.12c), we have set $D = 4-\epsilon$. Next, we evaluate the gamma matrix product in the numerator of the integrand:

$$\gamma^\mu(\not{p}(1-x)+m)\gamma_\mu = \gamma^\mu\gamma^\nu\gamma_\mu p_\nu(1-x) + m\gamma^\mu\gamma_\mu \tag{A.13a}$$

$$= (2-D)\not{p}(1-x) + Dm \tag{A.13b}$$

$$= (-2+\epsilon)\not{p}(1-x) + (4-\epsilon)m \tag{A.13c}$$

$$= 4m - 2\not{p}(1-x) + \epsilon\left(\not{p}(1-x) - m\right) \tag{A.13d}$$

In eq. (A.13b), we have used the identities in eq. (A.1i) and eq. (A.1j). We then expand the integrand in powers of ϵ to obtain:

$$i\Sigma(p) = \frac{-i\mu^\epsilon e^2}{(4\pi)^{2-\epsilon/2}} \int_0^1 dx \left[4m - 2\not{p}(1-x) + \epsilon\left(\not{p}(1-x) - m\right)\right] \Gamma(\epsilon/2) \left(-m^2 x + p^2 x(1-x)\right)^{-\epsilon/2} (-1)^{-\epsilon/2} \tag{A.14a}$$

$$= \frac{-ie^2}{(4\pi)^2} \int_0^1 dx \left[4m - 2\not{p}(1-x) + \epsilon\left(\not{p}(1-x) - m\right)\right] \left[\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right] \tag{A.14b}$$

$$\times \left[1 - \frac{\epsilon}{2}\log\left(\frac{-m^2 x + p^2 x(1-x)}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2)\right]\left[1 + \mathcal{O}(\epsilon^2)\right]$$

$$= \frac{-ie^2}{(4\pi)^2} \int_0^1 dx \Bigg\{ \left[4m - 2\not{p}(1-x)\right] \left[\frac{2}{\epsilon} - \gamma_E - \log\left(\frac{-m^2 x + p^2 x(1-x)}{4\pi\mu^2}\right)\right] \tag{A.14c}$$

$$+ 2\left(\not{p}(1-x) - m\right) + \mathcal{O}(\epsilon) \Bigg\}$$

$$= \frac{-ie^2}{16\pi^2} \left[\int_0^1 dx \left\{ \left[4m - 2\not{p}(1-x)\right] \left[\frac{2}{\epsilon} - \gamma_E\right] + 2\left(\not{p}(1-x) - m\right) \right\} \right. \tag{A.14d}$$

$$\left. - \int_0^1 dx \left\{ \left[4m - 2\not{p}(1-x)\right] \log\left(\frac{-m^2 x + p^2 x(1-x)}{4\pi\mu^2}\right) \right\} \right] + \mathcal{O}(\epsilon) \tag{A.14e}$$

$$= \frac{-ie^2}{16\pi^2} \left[\left\{ \left[4m - \not{p}\right] \left[\frac{2}{\epsilon} - \gamma_E\right] + 2\left(\not{p}/2 - m\right) \right\} \right. \tag{A.14f}$$

$$\left. - \int_0^1 dx \left\{ \left[4m - 2\not{p}(1-x)\right] \log\left(\frac{-m^2 x + p^2 x(1-x)}{4\pi\mu^2}\right) \right\} \right] + \mathcal{O}(\epsilon) \tag{A.14g}$$

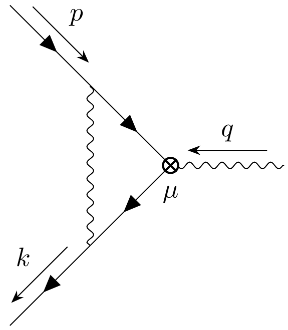
$$= \frac{ie^2}{8\pi^2\epsilon}\left(\not{p} - 4m\right) + \frac{ie^2}{16\pi^2} \left[2m(2\gamma_E + 1) - \not{p}(1 + \gamma_E) \right. \tag{A.14h}$$

$$\left. - 2\int_0^1 dx \left\{ \left[\not{p}(1-x) - 2m\right] \log\left(\frac{-m^2 x + p^2 x(1-x)}{4\pi\mu^2}\right) \right\} \right] + \mathcal{O}(\epsilon). \tag{A.14i}$$

Our result in eq. (A.14h) agrees with the result in eq. (3.1d).

A.3 QED Vertex Correction

The final divergent one-loop diagram we need to calculate is the QED vertex correction, which is given by the following diagram



$$\equiv -i\mu^{\epsilon/2}e\Lambda^{\mu}$$

Using the Feynman rules, we can write down the Feynman integral for the QED vertex correction as:

$$-i\mu^{\epsilon/2}e\Lambda^{\mu} = \left(-i\mu^{\epsilon/2}e\right)^3 \int \frac{d^D\ell}{(2\pi)^D}\gamma^{\nu}\frac{i(\not{k}-\not{\ell}+m)}{(k-\ell)^2-m^2+i0^+}\gamma^{\mu}\frac{i(\not{p}-\not{\ell}+m)}{(p-\ell)^2-m^2+i0^+}\gamma_{\rho}\frac{-ig_{\nu\rho}}{\ell^2+i0^+} \tag{A.15a}$$

$$= -\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\frac{\gamma^{\nu}(\not{k}-\not{\ell}+m)\gamma^{\mu}(\not{p}-\not{\ell}+m)\gamma_{\nu}}{\big((k-\ell)^2-m^2+i0^+\big)\big((p-\ell)^2-m^2+i0^+\big)\big(\ell^2+i0^+\big)}. \tag{A.15b}$$

Next, we use the Feynman parametrization identity in eq. (A.1b) to combine the denominators of the integrand:

$$-i\mu^{\epsilon/2}e\Lambda^{\mu} = -2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\gamma^{\nu}(\not{k}-\not{\ell}+m)\gamma^{\mu}(\not{p}-\not{\ell}+m)\gamma_{\nu} \tag{A.16a}$$

$$\times\left[x\big((p-\ell)^2-m^2\big)+y\big((k-\ell)^2-m^2\big)+(1-x-y)\ell^2+i0^+\right]^{-3}$$

$$= -2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\gamma^{\nu}(\not{k}-\not{\ell}+m)\gamma^{\mu}(\not{p}-\not{\ell}+m)\gamma_{\nu}}{\big((\ell-px-ky)^2-\Delta+i0^+\big)^3} \tag{A.16b}$$

$$= -2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\gamma^{\nu}(\not{k}(1-y)-\not{p}x-\not{\ell}+m)\gamma^{\mu}(\not{p}(1-x)-\not{k}y-\not{\ell}+m)\gamma_{\nu}}{\big(\ell^2-\Delta+i0^+\big)^3} \tag{A.16c}$$

In eq. (A.16b), we have defined $\Delta = m^2(x+y) - p^2x(1-x) - k^2y(1-y) + 2p\cdot kxy$; in eq. (A.16c), we have shifted the integration variable $\ell \to \ell + px + ky$. We now turn our attention to the numerator of the integrand in eq. (A.16c). In doing so we find the following:

$$\gamma^{\nu}(\overbrace{\not{k}(1-y)-\not{p}x}^{\equiv\not{Q}}-\not{\ell}+m)\gamma^{\mu}(\overbrace{\not{p}(1-x)-\not{k}y}^{\not{V}}-\not{\ell}+m)\gamma_{\nu} = \gamma^{\nu}\not{\ell}\gamma^{\mu}\not{\ell}\gamma_{\nu}+\gamma^{\nu}(\not{V}+m)\gamma^{\mu}(\not{Q}+m)\gamma_{\nu} \tag{A.17a}$$

$$+\ 2 \text{ terms linear in } \ell.$$

The linear terms in ℓ vanish upon integration over the symmetric ℓ domain. We then have

$$-i\mu^{\epsilon/2}e\Lambda^{\mu} = -2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\gamma^{\nu}\not{\ell}\gamma^{\mu}\not{\ell}\gamma_{\nu}}{\big(\ell^2-\Delta+i0^+\big)^3} \tag{A.18a}$$

$$-\ 2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\gamma^{\nu}(\not{V}+m)\gamma^{\mu}(\not{Q}+m)\gamma_{\nu}}{\big(\ell^2-\Delta+i0^+\big)^3}$$

$$= -i\mu^{\epsilon/2}e\Lambda^{\mu}_{(1)} - i\mu^{\epsilon/2}e\Lambda^{\mu}_{(2)} \tag{A.18b}$$

The terms $\Lambda^{\mu}_{(1)}$ and $\Lambda^{\mu}_{(2)}$ are infinite and finite, respectively. We will first evaluate the infinite term $\Lambda^{\mu}_{(1)}$:

$$-i\mu^{\epsilon/2}e\Lambda^{\mu}_{(1)} = -2\mu^{3\epsilon/2}e^3\gamma^{\nu}\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma_{\nu}\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\ell_{\sigma}\ell_{\rho}}{\big(\ell^2-\Delta+i0^+\big)^3}. \tag{A.19a}$$

Note that the integral above is only non-zero when $\sigma = \rho$. Thus, we can write;

$$-i\mu^{\epsilon/2}e\Lambda^\mu_{(1)} = -2\mu^{3\epsilon/2}e^2\gamma^\nu\gamma^\sigma\gamma^\mu\gamma_\sigma\gamma_\nu \int \frac{d^D\ell}{(2\pi)^D}\int_0^1 dx \int_0^{1-x} dy \frac{\ell^2}{\left(\ell^2 - \Delta + i0^+\right)^3} \tag{A.20a}$$

$$= -2\mu^{3\epsilon/2}e^2\gamma^\nu\gamma^\sigma\gamma^\mu\gamma_\sigma\gamma_\nu \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D\ell_E}{(2\pi)^D}\frac{i\ell_E^2}{\left(\ell_E^2+\Delta\right)^3} \tag{A.20b}$$

$$= -2i\mu^{3\epsilon/2}e^3\gamma^\nu\gamma^\sigma\gamma^\mu\gamma_\sigma\gamma_\nu \int_0^1 dx \int_0^{1-x} dy\, \frac{1}{2}\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}\Gamma(3)}\Delta^{D/2-2} \tag{A.20c}$$

$$= \frac{-i\mu^{3\epsilon/2}e^3}{2(4\pi)^{2-\epsilon/2}}(2-D)^2\gamma^\mu \int_0^1 dx \int_0^{1-x} dy\, \Gamma(\epsilon/2)\Delta^{-\epsilon/2} \tag{A.20d}$$

$$= -i\mu^{\epsilon/2}e\frac{e^2}{2(4\pi)^2}\gamma^\mu \int_0^1 dx \int_0^{1-x} dy\, \Gamma(\epsilon/2)\left(\frac{-\Delta}{4\pi\mu^2}\right)^{-\epsilon/2}(\epsilon-2)^2(-1)^{\epsilon/2}. \tag{A.20e}$$

In eq. (A.20b), we have performed a Wick rotation to Euclidean space, in eq. (A.20c), we have used the identity in eq. (A.1d), and in eq. (A.20d), we have used the identity in eq. (A.1k). Next, we expand the integrand in powers of ϵ to obtain:

$$-i\mu^{\epsilon/2}e\Lambda^\mu_{(1)} = -i\mu^{\epsilon/2}e\frac{e^2}{2(4\pi)^2}\gamma^\mu \int_0^1 dx \int_0^{1-x} dy \left[\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right] \tag{A.21a}$$

$$\times\left[1 - \frac{\epsilon}{2}\log\left(\frac{-\Delta}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2)\right]\left[4 - 4\epsilon + \mathcal{O}(\epsilon^2)\right]\left[1+\mathcal{O}(\epsilon^2)\right] \tag{A.21b}$$

$$= -i\mu^{\epsilon/2}e\frac{e^2}{2(4\pi)^2}\gamma^\mu \int_0^1 dx \int_0^{1-x} dy \left[\frac{8}{\epsilon} - 4\gamma_E - 8 - 4\log\left(\frac{-\Delta}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{A.21c}$$

$$= -i\mu^{\epsilon/2}e\Bigg\{\frac{e^2}{8\pi^2\epsilon} \tag{A.21d}$$

$$- \frac{e^2}{16\pi^2}\left[\gamma_E + 2 + 2\int_0^1 dx \int_0^{1-x} dy\, \log\left(\frac{-m^2(x+y) + p^2x(1-x) + k^2y(1-y) - 2p\cdot kxy}{4\pi\mu^2}\right)\right]\Bigg\}$$

$$+ \mathcal{O}(\epsilon).$$

We now turn our attention to the finite term $\Lambda^{\mu}_{(2)}$:

$$-i\mu^{\epsilon/2}e\Lambda^{\mu}_{(2)} = -2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\gamma^{\nu}(\not{\mathcal{P}}+m)\gamma^{\mu}(\not{\mathcal{Q}}+m)\gamma_{\nu}}{(\ell^2-\Delta+i0^+)^3} \tag{A.22a}$$

$$= -2\mu^{3\epsilon/2}e^3\int\frac{d^D\ell}{(2\pi)^D}\int_0^1 dx\int_0^{1-x} dy\frac{\gamma^{\nu}(\not{p}(1-x)-\not{k}y+m)\gamma^{\mu}(\not{k}(1-y)-\not{p}x+m)\gamma_{\nu}}{(\ell^2-\Delta+i0^+)^3} \tag{A.22b}$$

$$= -2\mu^{3\epsilon/2}e^3\int_0^1 dx\int_0^{1-x} dy\int\frac{d^D\ell_E}{(2\pi)^D}\frac{(-i)\gamma^{\nu}(\not{p}(1-x)-\not{k}y+m)\gamma^{\mu}(\not{k}(1-y)-\not{p}x+m)\gamma_{\nu}}{(\ell_E^2+\Delta+i0^+)^3} \tag{A.22c}$$

$$= -2i\mu^{3\epsilon/2}e^3\int_0^1 dx\int_0^{1-x} dy\;\gamma^{\nu}(\not{p}(1-x)-\not{k}y+m)\gamma^{\mu}(\not{k}(1-y)-\not{p}x+m)\gamma_{\nu} \times\frac{\Delta^{D/2-3}\Gamma(3-D/2)}{(4\pi)^{D/2}\Gamma(3)} \tag{A.22d}$$

$$= -i\mu^{\epsilon/2}e\frac{e^2\mu^{\epsilon}}{(4\pi)^{2-\epsilon/2}}\int_0^1 dx\int_0^{1-x} dy\;\gamma^{\nu}(\not{p}(1-x)-\not{k}y+m)\gamma^{\mu}(\not{k}(1-y)-\not{p}x+m)\gamma_{\nu} \times\Delta^{\epsilon/2-1}\Gamma(1-\epsilon/2) \tag{A.22e}$$

$$= -i\mu^{\epsilon/2}e\frac{e^2}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\;\frac{\gamma^{\nu}(\not{p}(1-x)-\not{k}y+m)\gamma^{\mu}(\not{k}(1-y)-\not{p}x+m)\gamma_{\nu}}{\Delta}+\mathcal{O}(\epsilon) \tag{A.22f}$$

Gathering the results for $\Lambda^{\mu}_{(1)}$ and $\Lambda^{\mu}_{(2)}$, we find that the QED vertex correction is given by:

$$-i\mu^{\epsilon/2}e\Lambda^{\mu} = -i\mu^{\epsilon/2}e\Bigg\{\frac{e^2}{8\pi^2\epsilon} \tag{A.23}$$

$$-\frac{e^2}{16\pi^2}\left[\gamma_E+2+2\int_0^1 dx\int_0^{1-x} dy\;\log\left(\frac{-m^2(x+y)+p^2x(1-x)+k^2y(1-y)-2p\cdot kxy}{4\pi\mu^2}\right)\right]$$

$$+\int_0^1 dx\int_0^{1-x} dy\;\frac{\gamma^{\nu}(\not{p}(1-x)-\not{k}y+m)\gamma^{\mu}(\not{k}(1-y)-\not{p}x+m)\gamma_{\nu}}{m^2(x+y)-p^2x(1-x)-k^2y(1-y)+2p\cdot kxy}+\mathcal{O}(\epsilon)\Bigg\}. \tag{A.24}$$

Which agrees with the result in eq. (3.1f).