String Theory Project

BRST Quantization in String Theory

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Abstract

This project explores the BRST formalism, developed by Carlo Becchi, Alain Rouet, Raymond Stora, and Igor Tyutin, with the aim of applying it to String Theory. In Section 1, we examine the mathematical challenges inherent in theories with gauge symmetry and address these challenges using the Faddeev-Popov method. Building on this, we introduce the BRST formalism, a powerful approach to quantizing theories with gauge symmetry. In Section 2, we provide an example of the BRST formalism in action by applying it to a relativistic point particle.

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1

BRST in String Theory

The BRST (Becchi-Rouet-Stora-Tyutin) quantization procedure is a technique used to quantize classical fields with gauge symmetry. Gauge symmetries in field theories present unique challenges, as propagators are not well-defined unless the theory is *gauge fixed* [1]. Additionally, gauge theories contain non-physical degrees of freedom, which can be problematic in the path integral formulation of quantum field theory.

To address these issues, the Faddeev-Popov method is employed. This method not only imposes gauge fixing on the theory but also mitigates the effects of non-physical degrees of freedom by introducing ghost fields. The result is an effective Lagrangian that includes both a gauge-fixing term and a term with ghost particles.

The BRST quantization process can then be applied to this effective Lagrangian. By leveraging the symmetries of the Lagrangian, the BRST procedure produces a conserved charge, which acts as a powerful tool to partition the Hilbert space of states. into physical and non-physical states.

1.1 *Faddeev-Popov Method*

In this section, we will provide a sketch of the Faddeev-Popov method [2].

Suppose we fix a particular set of reference fields $\{X_i^\mu\}$ such that each X_i^μ is distinct from every other field X_j^μ with $j \neq i$ in the sense that there is no gauge transformation connecting X_i^μ to X_j^μ. Now, for any arbitrary field $\bar{X}^\mu$ in the space of all field configurations, we may find a particular X_i^μ along with a gauge transformation U, such that acting the gauge transformation on X^μ will get us back to $\bar{X}_i^\mu$. Schematically we can then write a path integral as

$$\int \mathcal{D}[\bar{X}^\mu] e^{iS} \sim \int \mathcal{D}[X^\mu] \int \mathcal{D}U e^{iS} \tag{1}$$

where in the measure on the right we integrate over all fields including, those related by gauge transformations; on the left we only integrate over the reference fields in the set $\{X_i^\mu\}$ and then separately integrate over the gauge transformations. The issue at hand is that there are infinitely many gauge transformations U and thus the integral over these transformations will yield infinite contributions to our path integrals.

To circumvent this issue, one can attempt to *factor out* the integration of the fields that are related by a gauge transformation. This can be achieved by using the Faddeev-Popov method. Which in summary, does exactly this and factors out the *over-integration* by introducing a gauge fixing term and simultaneously introducing ghost fields into the original Lagrangian; a path integral will now take the following form

$$Z \propto \int \mathcal{D}[X_\mu, c, b, B] \exp\left\{\int d^4x \left(\mathscr{L} + \mathscr{L}_{GF} + \mathscr{L}_G\right)\right\} \tag{2}$$

where $\mathscr{L}$ is the original Lagrangian with the gauge symmetry while c and b are Grassman ghost fields and B is a Bosonic ghost field. The $\mathscr{L}_{GF}$ term arises from explicitly fixing a gauge and the $\mathscr{L}_G$ term contains the ghost fields. Together the three terms form an effective Lagrangian which we shall refer to as the BRST Lagrangian, *i.e.*

$$\mathscr{L}_{BRST} = \mathscr{L} + \mathscr{L}_{GF} + \mathscr{L}_G \tag{3}$$

1.2 *The BRST Transformation*

In this section, we will develop the formalism used in the BRST quantization technique in a general manner. In section 2, we will apply the formalism to a specific example: the point particle.

Let us begin by considering a set of fields $\{\phi_i(\sigma)\}$ that would generically make up a path integral in String Theory. For example, the $\phi_i(\sigma)$ fields could be composed of the target space coordinates $X^\mu(\sigma)$ and the metric $g_{\alpha\beta}(\sigma)$. In our generic path integral, we will find a Lagrangian $\mathscr{L}$. Now we will suppose that $\mathscr{L}$ has embedded within it some gauge symmetry of the form

$$\omega^\alpha(\sigma)\delta_\alpha \tag{4}$$

where the ω parameter is real and the α index may be discrete or continuous. We will find that the gauge transformations will form an algebra with commutation relations of the form

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma \tag{5}$$

where $f^\gamma_{\alpha\beta}$ are the structure constants of the algebra.

The gauge symmetry in $\mathscr{L}$ will allow us to impose a gauge fixing condition, which will be denoted here as

$$F^a(\phi) = 0 \tag{6}$$

where the index a specifies the particular components of the gauge fixing condition. For example, in Quantum Electrodynamics we could have A^μ for ϕ and under the Lorenz gauge, F would just be $\partial_\mu A^\mu$ (the a index would be superfluous here).

With this in mind, we can make use of the Faddeev-Poppov method to introduce two Grassman (anti-commuting) ghost fields b and c and a Bosonic (commuting) ghost field to *eat* the non-physical degrees of freedom; we can then write down a new effective Lagrangian for our theory, which as mentioned in eq. (3) will take the form

$$\mathscr{L}_{BRST} = \mathscr{L} + \mathscr{L}_{GF} + \mathscr{L}_G \tag{7}$$

The gauge fixing Lagrangian $\mathscr{L}_{GF}$ will be of the form

$$\mathscr{L}_{GF} = -iB_a F^a(\phi) \tag{8}$$

where the B is the Bosonic field and is related to the particular choice of gauge. The ghost Lagrangian

$\mathscr{L}_G$ will be of the following form

$$\mathscr{L}_G = b_a c^\alpha \delta_\alpha F^a(\phi) \tag{9}$$

1.3 Symmetries of the BRST Lagrangian

The BRST Lagrangian constructed above possesses an important symmetry. In particular, we will find that $\delta_B(\mathscr{L}_{BRST}) = 0$ where δ_B is the BRST variation; δ_B acts on the fields as follows

$$\delta_{\mathrm{B}}\phi_i = -i\epsilon c^\alpha \delta_\alpha \phi_i \tag{10a}$$
$$\delta_{\mathrm{B}} B_a = 0 \tag{10b}$$
$$\delta_{\mathrm{B}} b_a = \epsilon B_a \tag{10c}$$
$$\delta_{\mathrm{B}} c^\alpha = \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma \tag{10d}$$

where ϵ is a real parameter and δ_B is a variation which defines the transformations in eq. (10) and are known collectively as the BRST transformation.

Here we show that $\delta_B(\mathscr{L}_{BRST}) = 0$ explicitly. First note that the original Lagrangian $\mathscr{L}$ is invariant separately from the $\mathscr{L}_{GF}$ and $\mathscr{L}_G$ Lagrangians. To see this, we act the BRST variation on $\mathscr{L}$ to get

$$\delta_B(\mathscr{L}) = \delta_B\left(\mathscr{L}\left[\phi_i\right]\right) \tag{11a}$$
$$= \frac{\delta \mathscr{L}}{\delta \phi_i}\delta_B(\phi_i) \tag{11b}$$
$$= \frac{\delta \mathscr{L}}{\delta \phi_i}\left(-i\varepsilon c^\alpha \delta_\alpha \phi_i\right) \tag{11c}$$
$$= -i\varepsilon c^\alpha \frac{\delta \mathscr{L}}{\delta \phi_i}\delta_\alpha \phi_i \tag{11d}$$
$$= -i\varepsilon c^\alpha \delta_\alpha \mathscr{L} \tag{11e}$$
$$= 0 \tag{11f}$$

the last line follows since $\mathscr{L}$ is invariant under gauge symmetries of the form of eq. (4). Now the $\mathscr{L}_{GF}$ term transforms as follows

$$\delta_B\left(\mathscr{L}_{GF}\right) = \delta_B\left(-iB_a F^a\right) \tag{12a}$$
$$= -i\left[\delta_B\left(B_a\right)F^a + B_a \delta_B\left(F^a\right)\right] \tag{12b}$$
$$= -i\left[0 + B_a \delta_B\left(F^a\right)\right] \tag{12c}$$
$$= -iB_a \delta_B\left(F^a\right) \tag{12d}$$
$$= -iB_a \frac{\delta F^a}{\delta \phi_i}\delta_B \phi_i \tag{12e}$$
$$= -iB_a \frac{\delta F^a}{\delta \phi_i}\left(-i\varepsilon c^\alpha \delta_a \phi_i\right) \tag{12f}$$
$$= -\varepsilon B_a c^\alpha \delta_\alpha \phi_i \frac{\delta F^a}{\delta \phi_i} \tag{12g}$$
$$= -\varepsilon B_a c^\alpha \delta_\alpha F^a \tag{12h}$$

The BRST variation on the $\mathscr{L}_G$ term is somewhat more involved, we will have to break this calculation up to into a few steps.

$$\delta_B(\mathscr{L}_G) = \delta_B(b_a c^\alpha \delta_\alpha F^a) \tag{13a}$$
$$= \delta_B(b_a)\, c^\alpha \delta_\alpha F^a + b_a \delta_B(c^\alpha)\, \delta_\alpha F^a + b_a c^\alpha \delta_B(\delta_\alpha F^a) \tag{13b}$$
$$= \varepsilon B_a c^\alpha \delta_\alpha F^a + b_a \left(i/2\varepsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right) \delta_\alpha F^a + b_a c^\alpha \delta_\alpha(\delta_B F^a) \tag{13c}$$

The first term in eq. (13c) will immediately cancel the $\delta_B(\mathscr{L}_{GF})$ contribution. The second term in eq. (13c) can be rewritten as

$$b_a\left((i/2)\varepsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right)\delta_\alpha F^a = (i/2) b_a \varepsilon c^\beta c^\gamma f^\alpha_{\beta\gamma} \delta_\alpha F^a \tag{14a}$$
$$= (i/2) b_a \varepsilon c^\beta c^\gamma \left[\delta_\beta, \delta_\gamma\right] F^a \tag{14b}$$

where in eq. (14b) we made use of the commutation relations in eq. (5). The commutator can be simplified as follows

$$c^\beta c^\gamma\left[\delta_\beta, \delta_\gamma\right] = c^\beta c^\gamma \delta_\beta \delta_\gamma - c^\beta c^\gamma \delta_\gamma \delta_\beta \tag{15a}$$
$$= c^\beta c^\gamma \delta_\beta \delta_\gamma - c^\gamma c^\beta \delta_\beta \delta_\gamma \tag{15b}$$
$$= c^\beta c^\gamma \delta_\beta \delta_\gamma + c^\beta c^\gamma \delta_\beta \delta_\gamma \tag{15c}$$
$$= 2 c^\beta c^\gamma \delta_\beta \delta_\gamma \tag{15d}$$

where in eq. (15b) we just relabeled the γ and β indices, while in eq. (15c) we made use of the anti-commuting nature of the c fields. Putting all of this together, the second term in eq. (13c) becomes

$$b_a\left((i/2)\varepsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\right)\delta_\alpha F^a = i b_a \varepsilon c^\beta c^\gamma \delta_\beta \delta_\gamma F^a \tag{16}$$

Lastly, the third term in eq. (13c) can also be simplified

$$b_a c^\alpha \delta_\alpha(\delta_B F^a) = b_a c^\alpha \delta_B(\delta_\alpha F^a) \tag{17a}$$
$$= b_a c^\alpha \frac{\delta}{\delta \phi_i}(\delta_\alpha F^a)\, \delta_B(\phi_i) \tag{17b}$$
$$= b_a c^\alpha \frac{\delta}{\delta \phi_i}(\delta_\alpha F^a)\left(-i\varepsilon c^\beta \delta_\beta \phi_i\right) \tag{17c}$$
$$= -i\varepsilon b_a c^\alpha c^\beta \frac{\delta}{\delta \phi_i}(\delta_\alpha F^a)\, \delta_\beta \phi_i \tag{17d}$$
$$= -i\varepsilon b_a c^\alpha c^\beta \delta_\beta \delta_\alpha F^a \tag{17e}$$

So the second and third terms eq. (13c) cancel with each other. This then leaves the total variation on $\mathscr{L}_{BRST}$ as zero, which is what we wanted to prove.

1.4 *The Conserved Charge and Physical States*

We showed that the BRST Lagrangian is invariant under the BRST transformation, so by Noether's theorem we expect there to be a conserved charge. The conserved charge associated with the BRST transformation (denoted here as Q_B) will have some rather surprising properties. We will find that the conserved charge actually partitions the Hilbert space of states into physical and non-physical states.

To understand how Q_B partitions the Hilbert space, we shall consider a small change δF in the gauge fixing condition, then we will find that an arbitrary correlator $\langle f|i\rangle$ composed of arbitrary states $|i\rangle$ and $|f\rangle$ will vary as follows[1]

$$\begin{aligned}
\varepsilon\delta\langle f \mid i\rangle &= \varepsilon\left[\langle f \mid i\rangle_{F+\delta F} - \langle\delta \mid i\rangle_F\right] && \text{(18a)}\\
&= \varepsilon\left[\int \mathcal{D}[\Omega] e^{-S-S_{GF}(F+\delta F)-S_G(F+\delta F)} - \langle f \mid i\rangle_F\right] && \text{(18b)}\\
&= \varepsilon\left[\int \mathcal{D}[\Omega] \exp\left\{-\int d\tau[\mathscr{L} + \mathscr{L}_{GF}(F+\delta F) + \mathscr{L}_G(F+\delta F)]\right\} - \langle f \mid i\rangle_F\right] && \text{(18c)}\\
&= \varepsilon\left[\int \mathcal{D}[\Omega] e^{-S-S_{GF}(F)-S_G(F)} \exp\left\{-\int d\tau\left[-iB_a\left(\delta F^a\right) + b_a c^\alpha \delta_\alpha\left(\delta F^a\right)\right]\right\} - \langle f \mid i\rangle_F\right] && \text{(18d)}\\
&= \varepsilon\left[\int \mathcal{D}[\Omega] e^{-S-S_{GF}(F)-S_G(F)}\left(1 - \left[-iB_a\left(\delta F^a\right) + b_a c^\alpha \delta_\alpha\left(\delta F^a\right)\right] + \mathcal{O}(\delta F)^2\right) - \langle\delta \mid i\rangle_F\right] && \text{(18e)}\\
&= \varepsilon\int \mathcal{D}[\Omega] e^{-S-S_{GF}(F)-S_G(F)}\left[iB_a\left(\delta F^a\right) - b_a c^\alpha \delta_\alpha\left(\delta F^a\right)\right] && \text{(18f)}
\end{aligned}$$

where in eq. (18d) we used the linearity of $\mathscr{L}_{GF}$ and $\mathscr{L}_G$ in their F dependence, while in eq. (18e) we expanded the exponential to first order. We can now rewrite the last two terms in eq. (18f) as follows. Firstly, eq. (10c) implies that

$$\begin{aligned}
\delta_{\mathrm{B}} b_a &= \epsilon B_a && \text{(19a)}\\
\implies iB_a\delta F^a &= i\varepsilon^{-1}(\delta_B b_a)\delta F^a && \text{(19b)}
\end{aligned}$$

Now, the key step is to realize that F is a function of the fields ϕ, this will allow us to use the chain rule on the second term to get

$$\begin{aligned}
\delta_B\delta F^a\left(\phi_i\right) &= \delta\left(\delta_B F^a\left(\phi_i\right)\right) && \text{(20a)}\\
&= \delta\left(\frac{\delta F^a}{\delta\phi_i}\delta_B\phi_i\right) && \text{(20b)}\\
&= \delta\left(\frac{\partial F^a}{\delta\phi_i}\left[-i\varepsilon c^\alpha\delta_\alpha\phi_i\right]\right) && \text{(20c)}\\
&= \delta\left(-i\varepsilon c^\alpha\delta_\alpha F^a\right) && \text{(20d)}\\
&= -i\varepsilon c^\alpha\delta_\alpha\delta F^a && \text{(20e)}\\
\implies b_a c^\alpha\delta_\alpha(\delta F^\alpha) &= -i\varepsilon^{-1} b_a(\delta F^a) && \text{(20f)}
\end{aligned}$$

where in eq. (20c) we made use of eq. (10a). Gathering the results we have obtained for eq. (19) and eq. (20), we can return to the calculation for $\varepsilon\delta\langle f \mid i\rangle$ and obtain

$$\begin{aligned}
\epsilon\delta\langle f|i\rangle &= \epsilon(\langle f|i\rangle_{F+\delta F} - \langle f|i\rangle_F) && \text{(21a)}\\
&= \varepsilon\int \mathcal{D}[\Omega] e^{-S-S_{GF}(F)-S_G(F)}\left[i\varepsilon^{-1}(\delta_B b_a)\delta F^a + i\varepsilon^{-1} b_a(\delta F^a)\right] && \text{(21b)}
\end{aligned}$$

[1] the measure $D[\Omega]$ is a shorthand to integrate over all relevant fields in the path integral

$$= \varepsilon \int \mathcal{D}[\Omega] e^{-S-S_{GF}(F)-S_G(F)} \delta_{\mathrm{B}} \left(b_a \delta F^a\right) \tag{21c}$$

$$= i\langle f|\delta_{\mathrm{B}} \left(b_a \delta F^a\right)|i\rangle \tag{21d}$$

$$= -\epsilon\langle f| \left\{Q_{\mathrm{B}}, b_a \delta F^a\right\}|i\rangle \tag{21e}$$

where in the last line we made use of the following $\delta_B(\mathcal{A}) = i\epsilon\{Q_B, \mathcal{A}\}$ where Q_B is the BRST conserved current, this is true as we take the BRST variation to be a Hamiltonian vector field which is generated by the conserved charge [3].

Now for physical states, any change in the gauge condition F should leave the physical states unchanged, this follows since F is composed of the non-physical degrees of freedom in the theory. That is to say for any two physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$ we must have that

$$\langle \text{phys}|\text{phys}'\rangle_F \stackrel{!}{=} \langle \text{phys}|\text{phys}'\rangle_{F+\delta F} \tag{22}$$

If this is the case, then we must have that for any change δF in the gauge fixing

$$\langle \text{phys}| \left\{Q_{\mathrm{B}}, b_a \delta F^a\right\}|\text{phys}'\rangle = 0 \tag{23}$$

Now if we assume that Q_B is Hermitian, then we get the following result

$$Q_B\,|\text{phys}\rangle = Q_B\,|\text{phys}'\rangle = 0 \tag{24}$$

for all physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$.

So by exploiting the symmetry that arises in $\mathscr{L}_{BRST}$ we find, rather unexpectedly, that the conserved charge associated with the symmetry can be used as a powerful tool that allows us to distinguish between states that are physical and states that nonphysical. Neat!

The Conserved Charge is Nilpotent

The conserved charge has another important property: it is nilpotent, *i.e.* $Q_B^2 = 0$. To understand how the nilpotency of the conserved charge arises, one should realize that the charge should remain conserved at all times in order to move around the space of all gauge choices. Thus, the conserved charge should commute with the change in Hamiltonian, this implies the following

$$0 = \left[Q_{\mathrm{B}}, \left\{Q_{\mathrm{B}}, b_a \delta F^a\right\}\right] \tag{25a}$$

$$= \left[Q_{\mathrm{B}}, Q_{\mathrm{B}} b_a \delta F^a + Q_{\mathrm{B}} b_a \delta F^a\right] \tag{25b}$$

$$= Q_{\mathrm{B}}^2 b_a \delta F^a - Q_{\mathrm{B}} b_a \delta F^a Q_{\mathrm{B}} + Q_{\mathrm{B}} b_a \delta F^a Q_{\mathrm{B}} - b_a \delta F^a Q_{\mathrm{B}}^2 \tag{25c}$$

$$= \left[Q_{\mathrm{B}}^2, b_a \delta F^a\right] \tag{25d}$$

and inorder for $\left[Q_{\mathrm{B}}^2, b_a \delta F^a\right]$ to vanish for arbitrary variations in F, we must have that $Q_B = 0$.

1.5 Cohomology of the Conserved Charge

In the previous sections, we showed that the BSRT charge Q_B has the following two properties

1. $Q_B|\text{phys}\rangle = 0$ for all physical states $|\text{phys}\rangle$
2. Q_B is nilpotent, *i.e.*, Q_B^2 is zero as an operator

So for any state $|\chi\rangle$ in the Hilbert space $\mathscr{H}$, we must have the following

$$Q_B(Q_B|\chi\rangle) = 0 \tag{26}$$

and thus $Q_B|\chi\rangle$ is a physical state. Interestingly, $Q_B|\chi\rangle$ is orthogonal to every physical state (including itself).

$$\langle\text{phys}|(Q_B|\chi\rangle) = (\langle\text{phys}|Q_B)|\chi\rangle = 0 \tag{27}$$

where we have again assumed that Q_B is Hermitian. Let us refer to states of the form $Q_B|\chi\rangle$ as exact states. Now any two physical states that differ by an exact state, *i.e.*

$$|\text{phys}'\rangle = |\text{phys}\rangle + Q_B|\chi\rangle \tag{28}$$

will have the same inner products with all other physical states in the Hilbert space (by the orthogonality mentioned above). We can thus interpret any two physical states which differ by an exact state as physically equivalent. The true physical space (call it C) is thus seen as the set of equivalence classes, *i.e.*

$$C = \{[\psi_i] : i \in \mathcal{I}\}$$

where $[\psi_i]$ is the equivalence class defined as

$$[\psi_i] = \{|\phi\rangle : |\phi\rangle = |\psi_i\rangle + Q_B|\chi\rangle : Q_B|\psi_i\rangle = 0\}$$

The true physical space C is the *cohomology* of Q_B. The physical states are often referred to as *closed* states (exact states are a subset of the closed states).

2

BRST and the Relativistic Point Particle

In the previous section, we described how the BRST quantization technique can be applied to String Theory in a general formalism. In this section, we shall explore a particular example as an application of the BRST technique. We shall consider a relativistic point particle, which has the following einbein action [4]

$$S = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 + em^2\right) \tag{29}$$

where the $e = e(\tau)$ field is the einbein and the Lagrangian for the point particle is

$$\mathscr{L} = \frac{1}{2}\left(e^{-1}\dot{X}^2 + em^2\right) \tag{30}$$

The action in eq. (29) is invariant under the following set of transformations

$$\tau \mapsto \tilde{\tau}(\tau) \tag{31a}$$

$$X^\mu \mapsto \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau) \tag{31b}$$

$$e \mapsto \tilde{e}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}}e(\tau) \tag{31c}$$

The invariance of the action arises due to the Jacobian factor canceling with the factors that come along with the $\dot{\tilde{X}}^2$ and $\tilde{e}$ terms as we show here explicitly

$$\tilde{S} = \frac{1}{2}\int d\tilde{\tau}\left(\tilde{e}^{-1}\dot{\tilde{X}}^2 + \tilde{e}m^2\right) \tag{32a}$$

$$= \frac{1}{2}\int d\tau \frac{d\tilde{\tau}}{d\tau}\left\{\left(\frac{d\tau}{d\tilde{\tau}}e\right)^{-1}\left(\frac{d\tau}{d\tilde{\tau}}\right)^2\dot{X}^2 + \left(\frac{d\tau}{d\tilde{\tau}}e\right)m^2\right\} \tag{32b}$$

$$= \frac{1}{2}\int d\tau\left(e^{-1}\dot{X}^2 + em^2\right) \tag{32c}$$

$$= S \tag{32d}$$

So if we choose our coordinate transformation to take the following form

$$\tilde{\tau}(\tau) = \tau + \delta\tau \tag{33}$$

then the gauge symmetry of the einbein action is just the coordinate transformation $\delta\tau$. We can write down a basis for these coordinate transformations as

$$\delta_{\tau_i}\tau = \delta(\tau - \tau_i) \tag{34}$$

so that the variation only triggers when it is being applied at the same location as the coordinate on the world-line. The basis of transformations will then act infinitesimally on the X and e fields as follows

$$\delta_{\tau_i} X^\mu(\tau) = -\delta(\tau - \tau_i)[\partial_\tau X^\mu(\tau)] \tag{35a}$$
$$\delta_{\tau_i} e(\tau) = -\partial_\tau[\delta(\tau - \tau_i)e(\tau)] \tag{35b}$$

The structure constants mentioned in eq. (5) can be found by considering the following calculation

$$\begin{aligned}
[\delta\tau_i, \delta\tau_j]\, X^\mu(\tau) &= \delta\tau_i\delta\tau_j X^\mu - \delta\tau_j\delta\tau_i X^\mu && (36a)\\
&= \delta\tau_i\left(-\delta\left(\tau - \tau_j\right)\partial_\tau X^\mu\right) - \delta\tau_j\left(-\delta\left(\tau - \tau_i\right)\partial_\tau X^\mu\right) && (36b)\\
&= -\delta\left(\tau - \tau_j\right)\partial_\tau\delta\tau_i\left(X^\mu\right) + \delta\left(\tau - \tau_i\right)\partial_\tau\delta_{\tau_j}\left(X^\mu\right) && (36c)\\
&= -\delta\left(\tau - \tau_j\right)\partial_\tau\left(-\delta\left(\tau - \tau_i\right)\partial_\tau X^\mu\right) \\
&\quad + \delta\left(\tau - \tau_i\right)\partial_\tau\left(-\delta\left(\tau - \tau_j\right)\partial_\tau X^\mu\right) && (36d)\\
&= \delta\left(\tau - \tau_j\right)\left\{\left(\partial_\tau\delta\left(\tau - \tau_i\right)\right)\partial_\tau X^\mu + \delta\left(\tau - \tau_i\right)\left(\partial_\tau^2 X^\mu\right)\right\} \\
&\quad - \delta\left(\tau - \tau_i\right)\left\{\left(\partial_\tau\delta\left(\tau - \tau_j\right)\right)\partial_\tau X^\mu + \delta\left(\tau - \tau_j\right)\left(\partial_\tau^2 X^\mu\right)\right\} && (36e)\\
&= \left\{\delta\left(\tau - \tau_j\right)\left(\partial_\tau\delta\left(\tau - \tau_i\right)\right) - \delta\left(\tau - \tau_i\right)\left(\partial_\tau\delta\left(\tau - \tau_j\right)\right)\right\}\partial_\tau X^\mu && (36f)
\end{aligned}$$

In eq. (36b) and eq. (36d) we used eq. (35a). We now insert two nutritious factors of one in the form of integrals over deltas in each term of the last line to get

$$\begin{aligned}
[\delta\tau_i, \delta\tau_j]\, X^\mu(\tau) &= \int d\tau_k \delta(\tau_k - \tau)\left\{\delta\left(\tau_k - \tau_j\right)\left(\partial_{\tau_k}\delta\left(\tau_k - \tau_i\right)\right) - \delta\left(\tau_k - \tau_i\right)\left(\partial_{\tau_k}\delta\left(\tau_k - \tau_j\right)\right)\right\}\partial_\tau X^\mu && (37a)\\
&= \int d\tau_k\left\{\delta\left(\tau_k - \tau_j\right)\left(\partial_{\tau_k}\delta\left(\tau_k - \tau_i\right)\right) - \delta\left(\tau_k - \tau_i\right)\left(\partial_{\tau_k}\delta\left(\tau_k - \tau_j\right)\right)\right\}\left(-\delta_{\tau_k} X^\mu\right) && (37b)\\
&= \int d\tau_k f^{\tau_k}_{\tau_i\tau_j}\delta\tau_k X^\mu && (37c)
\end{aligned}$$

where in going from eq. (37b) to eq. (37c) we read off the structure constants as

$$f^{\tau_k}_{\tau_i\tau_j} = \delta\left(\tau_k - \tau_i\right)\left(\partial_{\tau_k}\delta\left(\tau_k - \tau_j\right)\right) - \delta\left(\tau_k - \tau_j\right)\left(\partial_{\tau_k}\delta\left(\tau_k - \tau_i\right)\right) \tag{38}$$

2.1 *BRST Transformation for the Point Particle*

The BRST transformation for the point particle has the form

$$\delta_{\mathrm{B}} X^{\mu} = i\epsilon c \dot{X}^{\mu} \tag{39a}$$
$$\delta_{\mathrm{B}} e = i\epsilon \dot{c} e \tag{39b}$$
$$\delta_{\mathrm{B}} B = 0 \tag{39c}$$
$$\delta_{\mathrm{B}} b = \epsilon B \tag{39d}$$
$$\delta_{\mathrm{B}} c = i\epsilon \partial_{\tau}[ce] \tag{39e}$$

Let us show how we arrived at eq. (39) from eq. (10) explicitly. For the BRST variation on the X^{μ} fields we have the following

$$\begin{aligned}\delta_B X^{\mu} &= -i\varepsilon \int d\tau_i c\left(\tau_i\right) \delta\tau_i X^{\mu} && (40a)\\ &= -i\varepsilon \int d\tau_i c\left(\tau_i\right) \left\{-\delta\left(\tau-\tau_i\right) \partial_{\tau} X^{\mu}\right\} && (40b)\\ &= i\varepsilon c(\tau) \partial_{\tau} X^{\mu} && (40c)\end{aligned}$$

While for the einbein field, we get

$$\begin{aligned}\delta_B e(\tau) &= -i\varepsilon \int d\tau_i c\left(\tau_i\right) \delta\tau_i e(\tau) && (41a)\\ &= -i\varepsilon \int d\tau_i c\left(\tau_i\right) \left\{-\partial_{\tau}\left[\delta\left(\tau-\tau_i\right) e(\tau)\right]\right\} && (41b)\\ &= i\varepsilon \int d\tau_i c\left(\tau_i\right) \left\{\left[\partial_{\tau} \delta\left(\tau-\tau_i\right)\right] e(\tau) + \delta\left(\tau-\tau_i\right)\left[\partial_{\tau} e(\tau)\right]\right\} && (41c)\\ &= i\varepsilon \int d\tau_i c\left(\tau_i\right) \left\{\left[-\partial_{\tau_i} \delta\left(\tau-\tau_i\right)\right] e(\tau) + \delta\left(\tau-\tau_i\right)\left[\partial_{\tau} e(\tau)\right]\right\} && (41d)\\ &= i\varepsilon \int d\tau_i \left\{c\left(\tau_i\right)\left[-\partial_{\tau_i} \delta\left(\tau-\tau_i\right)\right] e(\tau) + c\left(\tau_i\right) \delta\left(\tau-\tau_i\right)\left[\partial_{\tau} e(\tau)\right]\right\} && (41e)\\ &= i\varepsilon \int d\tau_i \left\{\left[\partial_{\tau_i} c\left(\tau_i\right)\right] \delta\left(\tau-\tau_i\right) e(\tau) + c\left(\tau_i\right) \delta\left(\tau-\tau_i\right)\left[\partial_{\tau} e(\tau)\right]\right\} && (41f)\\ &= i\varepsilon \left\{\left[\partial_{\tau} c\right] e + c\left[\partial_{\tau} e\right]\right\} && (41g)\\ &= i\varepsilon \partial_t [ce] && (41h)\end{aligned}$$

where in eq. (41d) we switched the sign on the derivative acting on the delta by switching the derivative from τ to τ_i and then in eq. (41f) we used integration by parts to get rid of the total derivative by noticing that $e = e(\tau) \neq e(\tau_i)$. The BSRT variations on the B and b fields are just trivially true. All that's left is the BRST variation acting on the c field, this is calculated as follows

$$\begin{aligned}\delta_B c &= \frac{i}{2} \varepsilon \int d\tau_i d\tau_j f^{\tau} \tau_i \tau_j c\left(\tau_i\right) c\left(\tau_j\right) && (42a)\\ &= \frac{i\varepsilon}{2} \int d\tau_i d\tau_j \left\{\delta\left(\tau-\tau_i\right) \partial_{\tau} \delta\left(\tau-\tau_j\right) - \delta\left(\tau-\tau_j\right) \partial_{\tau} \delta\left(\tau-\tau_i\right)\right\} c\left(\tau_i\right) c\left(\tau_j\right) && (42b)\end{aligned}$$

$$= \frac{i\varepsilon}{2}\left\{\int d\tau_j \partial\tau_i \delta\left(\tau_i-\tau_j\right) c(\tau) c\left(\tau_j\right)-\int d\tau_i \partial_{\tau_j} \delta\left(\tau_j-\tau_i\right) c\left(\tau_i\right) c(\tau)\right\} \tag{42c}$$

$$= i\varepsilon \int d\tau_j \partial\tau_i \delta\left(\tau_i-\tau_j\right) c(\tau) c\left(\tau_j\right) \tag{42d}$$

$$= -i\varepsilon \int d\tau_j \partial_{\tau_j} \delta\left(\tau_i-\tau_j\right) c(\tau) c\left(\tau_j\right) \tag{42e}$$

$$= i\varepsilon \int d\tau_j \partial_{\tau_j} \delta\left(\tau_i-\tau_j\right) c\left(\tau_j\right) c(\tau) \tag{42f}$$

$$= -i\varepsilon \int d\tau_j \delta\left(\tau_i-\tau_j\right) \partial_{\tau_j} c\left(\tau_j\right) c(\tau) \tag{42g}$$

$$= -i\varepsilon \dot{c} c \tag{42h}$$

$$= i\varepsilon c \dot{c} \tag{42i}$$

in eq. (42d) we commuted the $c(\tau_i)$ past the $c(\tau)$ field which bought us a minus sign, then switching the indices allowed us to move to eq. (42d). In eq. (42f) we switched the derivative on the delta which gave the additional minus sign, this then allowed us to do integration by parts in eq. (42g), the last sign flip in eq. (42g) came from commuting the c field with its derivative since both of which are Grassman odd.

We now impose our gauge fixing by choosing $e(\tau) = 1$ so that our gauge condition becomes

$$F(\epsilon) = 1 - \epsilon \tag{43}$$

Each of the terms in the BSRT Lagrangian ($\mathscr{L}_{BRST} = \mathscr{L} + \mathscr{L}_{GF} + \mathscr{L}_{G}$) can now be written down explicitly. We find the gauge fixed Lagrangian ($\mathscr{L}_{GF}$) for the point particle to be

$$\mathscr{L}_{GF} = iBF = iB(e-1) \tag{44}$$

while the ghost Lagrangian ($\mathscr{L}_{G}$) takes the following form

$$\mathscr{L}_G = \int d\tau_1 b\left(\tau_1\right) c(\tau) \delta_\tau F\left(\tau_1\right) \tag{45a}$$

$$= \int d\tau_1 b\left(\tau_1\right) c(\tau) \delta_\tau\left(1-e\left(\tau_1\right)\right) \tag{45b}$$

$$= \int d\tau_1 b\left(\tau_1\right) c(\tau) \partial_{\tau_1}\left[\delta\left(\tau_1-\tau\right) e\left(\tau_1\right)\right] \tag{45c}$$

$$= -\int d\tau_1 \quad \left[\partial_{\tau_1} b\left(\tau_1\right)\right] c(\tau) \delta\left(\tau_1-\tau\right) e\left(\tau_1\right) \tag{45d}$$

$$= -\dot{b} c e \tag{45e}$$

$$= -e \dot{b} c \tag{45f}$$

where in eq. (45c) we used eq. (35b) and in eq. (45d) we integrated by parts. Gathering our results, we get the following for the full BRST Lagrangian.

$$\mathscr{L}_{BRST} = \mathscr{L} + \mathscr{L}_{GF} + \mathscr{L}_G \tag{46a}$$

$$= \frac{1}{2}\left(e^{-1} \dot{X}^2 + e m^2\right) + iB(e-1) - e\dot{b}c \tag{46b}$$

We can find the B field in terms of the other fields by considering the equations of motion for the e field

$$0 = \frac{\partial \mathscr{L}_{BRST}}{\partial e} - \partial_\mu \left(\frac{\partial \mathscr{L}_{BRST}}{\partial(\partial_\mu e)} \right) \tag{47a}$$

$$= -\frac{1}{2}\left(e^{-2}\dot{X}^2 - m^2\right) + iB - \dot{b}c \tag{47b}$$

$$\implies B = -i\left(\frac{\dot{X}^2}{2e^2} - \frac{m^2}{2} + \dot{b}c\right) \tag{47c}$$

Returning to the BRST Lagrangian, we will find it convenient to integrate out the B field in eq. (2) and absorb it into an overall normalization constant. This is possible since the B field no longer appears explicitly in the path integral. With this, the BRST Lagrangian can be rewritten as

$$\mathscr{L}_{BRST} = \frac{1}{2}\left(\dot{X}^2 + m^2\right) - \dot{b}c \tag{48}$$

and the BRST transformations become

$$\delta_{\rm B} X^\mu = i\epsilon c \dot{X}^\mu \tag{49a}$$

$$\delta_{\rm B} b = -i\epsilon\left(\frac{1}{2}\dot{X}^2 - \frac{1}{2}m^2 + \dot{b}c\right) \tag{49b}$$

$$\delta_{\rm B} c = i\epsilon c\dot{c} \tag{49c}$$

where we have dropped the B field variation as it no longer appears in the path integral, we have also replaced the B field that appears in eq. (39d) with eq. (47)

Under these transformations, the BRST Lagrangian is invariant as we show here explicitly. The nilpotency of δ_B also carries over, we check this here.

2.2 *Conserved Charge*

In what follows we shall derive the conserved charge (Q_B) for the point particle. We will find that Q_B takes the following form

$$Q_B = \int d\tau c(\tau)\mathcal{H}_{BRST}(\tau) \tag{50}$$

where $\mathcal{H}_{BRST}$ is the Hamiltonian associated with the BRST Lagrangian. To see this, we start by finding the conjugate momenta to the fields. For the X fields we find

$$p^\mu = \frac{\partial \mathscr{L}_{BRST}}{\partial\left(\partial_t X_\mu\right)} \tag{51a}$$

$$= \frac{\partial \mathscr{L}_{BRST}}{\partial\left(-i\partial_\tau X_\mu\right)} \tag{51b}$$

$$= i\frac{\partial}{\partial\left(\partial \dot{X}_\mu\right)}\left\{\frac{1}{2}\left(\dot{X}^2 + m^2\right) - \dot{b}c\right\} \tag{51c}$$

$$= i\dot{X}^\mu = \partial_t X^\mu \tag{51d}$$

where we were careful to distinguish between Euclidean time τ and Minkowski time $t = -i\tau$. The c field is not dynamical, so its conjugate momentum is zero. For the b field we get

$$\begin{aligned}
\Pi_b &= \frac{\partial \mathscr{L}_{BRST}}{\partial\left(\partial_t b\right)} && \text{(52a)}\\
&= i\frac{\partial \mathscr{L}_{BRST}}{\partial\left(\partial_\tau b\right)} && \text{(52b)}\\
&= i\frac{\partial}{\partial \dot{b}}\left\{\frac{1}{2}\left(\dot{X}^2+m^2\right)-\dot{b}c\right\} && \text{(52c)}\\
&= -ic && \text{(52d)}
\end{aligned}$$

again we had to be careful with the Euclidean and Minkowski times. To find the Hamiltonian (with all the signs correct), it will be easier to work with the Lagrangian $\mathscr{L}^M_{BRST}$ from the action in Minkowski coordinates. To find $\mathscr{L}^M_{BRST}$ we first consider S^E_{BRST}, the action in Euclidean coordinates. This just takes the form

$$S^E_{BRST} = \int_{\tau_i}^{\tau_s} d\tau \quad \frac{1}{2}\left\{\partial_\tau X^\mu \partial_\tau X_\mu + m^2\right\} - \partial_\tau bc \tag{53}$$

upon the change of variables $\tau \to it$, this becomes

$$S^E_{BRST} = \int_{it_i}^{it_f} idt\frac{1}{2}\left\{-\partial_t X^\mu \partial_t X_\mu + m^2\right\} + i\partial_t bc \tag{54}$$

Now the exponential that would appear in path integrals for Euclidean coordinates is of the form

$$\begin{aligned}
e^{-S^E_{BRST}} &= \exp\left[-\int_{it_i}^{it_f} idt\left(\frac{1}{2}\left\{-\partial_t X^\mu \partial_t X_\mu + m^2\right\} + i\partial_t bc\right)\right] && \text{(55a)}\\
&= \exp\left[i\int_{it_i}^{it_f} dt\left(\frac{1}{2}\left\{\partial_t X^\mu \partial_t X_\mu - m^2\right\} - i\partial_t bc\right)\right] && \text{(55b)}
\end{aligned}$$

The exponential in Minkowski coordinates will take the form $e^{iS^M_{BRST}}$, and of course, this must be equal to the exponential we found for the Euclidean coordinates. So we can read off $\mathscr{L}^M_{BRST}$ as

$$\mathscr{L}^M_{BRST} = \frac{1}{2}\left\{\partial_t X^\mu \partial_t X_\mu - m^2\right\} - i\partial_t bc \tag{56}$$

We can now find the Hamiltonian through the Legendre transform of the Lagrangian

$$\begin{aligned}
\mathcal{H}_{BRST} &= \sum_i [\partial_t \phi_i] p_i - \mathscr{L}^M_{BRST} && \text{(57a)}\\
&= \partial_t X_\mu p^\mu + \partial_t b \Pi_b - \mathscr{L}^M_{BRST} && \text{(57b)}\\
&= \partial_t X^\mu \partial_t X^\mu + \partial_t b(ic) - \frac{1}{2}\left\{\partial_t X^\mu \partial_t X^\mu - m^2\right\} + i\partial_t bc && \text{(57c)}\\
&= \frac{1}{2}\left\{\partial_t X^\mu \partial_t X_\mu + m^2\right\} && \text{(57d)}\\
&= \frac{1}{2}\left\{p^2 + m^2\right\} && \text{(57e)}
\end{aligned}$$

Now the conserved current can be found by making use of the Noether procedure, in particular, we act the BRST variation on the Lagrangian and make use of the symmetry that the Lagrangian has with respect to the variation. Still working with the Minkowski coordinates, we perform the following calculations

$$\delta_B \mathscr{L}^M_{BRST} = \frac{\partial \mathscr{L}^M_{BRST}}{\partial \phi_i}\delta_B\phi_i + \frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t\phi_i\right)}\delta_B\left(\partial_t\phi_i\right) \tag{58a}$$

$$= \frac{\partial \mathscr{L}^M_{BRST}}{\partial \phi_i}\delta_B\phi_i + \frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t\phi_i\right)}\partial_t\left(\delta_B\phi_i\right) \tag{58b}$$

$$= \underbrace{\left[\frac{\partial \mathscr{L}^M_{BRST}}{\partial \phi_i} - \partial_t\left(\frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t\phi_i\right)}\right)\right]}_{\text{0 on the eq. of motion}}\delta_B\phi_i + \partial_t\left(\frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t\phi_i\right)}\delta_B\phi_i\right) \tag{58c}$$

$$= \partial_t\left(\frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t\phi_i\right)}\delta_B\phi_i\right) \tag{58d}$$

$$= \partial_t\left(\frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t X_\mu\right)}\delta_B X_\mu + \frac{\partial \mathscr{L}^M_{BRST}}{\partial\left(\partial_t b\right)}\delta_B b\right) \tag{58e}$$

$$= \partial_t\left(\partial_t X^\mu \delta_B X_\mu + -ic\delta_B b\right) \tag{58f}$$

$$= \partial_t\left(\partial_t X^\mu \underbrace{\varepsilon c\partial_t X_\mu}_{\text{eq. (49a)}} - ici\varepsilon\overbrace{\left[\frac{1}{2}\partial_t \chi^\mu\partial_t X_\mu + \frac{1}{2}m^2 + i\partial_t bc\right]}^{\text{eq. (49b)}}\right) \tag{58g}$$

$$= \partial_t\left(\frac{3}{2}\varepsilon c\partial_t X^\mu\partial_t X^\mu + \frac{c\varepsilon}{2}m^2 - i\varepsilon\partial t bc^2\right) \tag{58h}$$

$$= \varepsilon\partial_t\left(c\left\{\frac{3}{2}\partial_t X^\mu\partial_t X^\mu + \frac{m^2}{2}\right\}\right) \tag{58i}$$

Now acting the BRST variation on the Lagrangian directly, we find

$$\delta_B \mathscr{L}^M_{BRST} = \frac{1}{2}\left\{\delta_B\left(\partial_t X^\mu\partial_t X_\mu\right) - \delta_B\left(m^2\right)\right\} - i\delta_B\left(\partial_t bc\right) \tag{59a}$$

$$= \partial_t X^\mu\partial_t\delta_B X_\mu - i\partial_t\left(\delta_B b\right)c - i\left(\partial_t b\right)\delta_B c \tag{59b}$$

$$= \partial_t X^\mu\partial_t\underbrace{\left(\varepsilon c\partial_t X_\mu\right)}_{\text{eq. (49a)}} - i\partial_t\left(i\varepsilon\overbrace{\left\{\frac{1}{2}\partial_t X^\mu\partial_t X_\mu + \frac{m^2}{2} + i\partial_t bc\right\}}^{\text{eq. (49b)}}\right)c - i\partial_t b\underbrace{\varepsilon c\partial_t c}_{\text{eq. (49c)}} \tag{59c}$$

$$= \varepsilon c\partial_t X^\mu\partial_t^2 X_\mu + \varepsilon\partial_t c\partial_t X^\mu\partial_t X_\mu \varepsilon c\partial_t X^\mu\partial_t^2 X_\mu + i\varepsilon\partial_t^2 bcc + i\varepsilon\partial_{tb}\partial_t cc - i\varepsilon\partial_t b\partial_t cc \tag{59d}$$

$$= \varepsilon\left(2c\partial_t X^\mu\partial_t^2 X_\mu + \partial_t c\partial_t X^\mu\partial_t X_\mu\right) \tag{59e}$$

$$= \varepsilon\partial_t\left(c\partial_t X^\mu\partial_t X_\mu\right) \tag{59f}$$

where in eq. (59d) we used that $c^2 = 0$ as it is a Grassman variable. Gathering the results from eq. (58) and eq. (59) we get the following equality

$$\varepsilon\partial_t\left(c\partial_t X^\mu\partial_t X_\mu\right) = \varepsilon\partial_t\left(c\left\{\frac{3}{2}\partial_t X^\mu\partial_t X^\mu + \frac{m^2}{2}\right\}\right) \tag{60a}$$

$$\implies 0 = \partial_t \left\{ c \left\{ \frac{1}{2}\partial_t X^\mu \partial_t X^\mu + \frac{m^2}{2} \right\} \right\} \tag{60b}$$

$$= \partial_t \left\{ c\mathcal{H}_{BRST} \right\} \tag{60c}$$

Thus we find that our conserved charge (Q_B) is indeed just the integral over all τ of c times the Hamiltonian, as advertised in eq. (50).

2.3 *Physical States*

For the point particle, we have the ghost fields b and c which will generate a two-state system. Let the complete set of states in our system be denoted by $|k, \uparrow\rangle$ and $|k, \downarrow\rangle$, these states will have the following properties

$$p^\mu|k,\downarrow\rangle = k^\mu|k,\downarrow\rangle \qquad p^\mu|k,\uparrow\rangle = k^\mu|k,\uparrow\rangle \tag{61a}$$

$$b|k,\downarrow\rangle = 0 \qquad b|k,\uparrow\rangle = |k,\downarrow\rangle \tag{61b}$$

$$c|k,\downarrow\rangle = |k,\uparrow\rangle \qquad c|k,\uparrow\rangle = 0 \tag{61c}$$

We can see of the c and b fields (which have now been promoted to operators) as raising and lowering operators respectively. Acting the conserved charge on the $|k, \downarrow\rangle$ state we find

$$Q_B\,|k,\downarrow\rangle = c\mathcal{H}_{BRST}\,|k,\downarrow\rangle \tag{62a}$$

$$= \frac{c}{2}(p^2 + m^2)\,|k,\downarrow\rangle \tag{62b}$$

$$= \frac{1}{2}(k^2 + m^2)c\,|k,\downarrow\rangle \tag{62c}$$

$$= \frac{1}{2}(k^2 + m^2)\,|k,\uparrow\rangle \tag{62d}$$

where we have left the integration over τ implicit, while acting Q_B on $|k, \uparrow\rangle$ gives

$$Q_B\,|k,\uparrow\rangle = c\mathcal{H}_{BRST}\,|k,\uparrow\rangle \tag{63a}$$

$$= \frac{c}{2}(p^2 + m^2)\,|k,\uparrow\rangle \tag{63b}$$

$$= \frac{1}{2}(k^2 + m^2)c\,|k,\uparrow\rangle \tag{63c}$$

$$= 0 \tag{63d}$$

Recall that the physical states are the states that are annihilated by the conserved charge. We see that $|k, \downarrow\rangle$ will be a physical state if $k^2 + m^2 = 0$, that is if its momentum is on-shell.

While the state $|k, \uparrow\rangle$ is a physical state for all k. But there are two cases to consider here, namely if the momentum of the state $|k, \uparrow\rangle$ is on-shell or not. If k is on-shell we can safely say that $|k, \uparrow\rangle$ is a physical state with momentum k, no problem there. But if k is off-shell, things are a little more interesting and we have to consider the cohomology of Q_B. Firstly, we note that since $k^2 + m^2 \neq 0$, then by eq. (62) we have that

$$|k,\uparrow\rangle = \frac{2}{k^2 + m^2} Q_B\,|k,\downarrow\rangle = Q_B\,|\chi\rangle \tag{64}$$

where $|\chi\rangle = 2\,|k,\downarrow\rangle / (k^2 + m^2)$. So this implies that $|k,\uparrow\rangle$ is an exact state. Now consider the state $|q,\downarrow\rangle$ defined as

$$|q,\downarrow\rangle := |k,\uparrow\rangle + Q_B|y\rangle \tag{65}$$

where $|y\rangle$ is some element of our two-state Hilbert space. By construction, $Q_B\,|q,\downarrow\rangle = 0$ and thus $|q,\downarrow\rangle$ is a physical state. By definition $|q,\downarrow\rangle$ and $|k,\uparrow\rangle$ are physically equivalent as they form part of the same equivalence class. But (here's the interesting part) the momentum of the state $|q,\downarrow\rangle$ is actually on-shell.

$$0 \overset{!}{=} Q_B\,|q,\downarrow\rangle \tag{66a}$$

$$= \frac{1}{2}(q^2 + m^2)\,|q,\uparrow\rangle \tag{66b}$$

$$\implies q^2 + m^2 = 0 \tag{66c}$$

So we can conclude that if k is off-shell, then $|k,\uparrow\rangle$ is physically equivalent to the state $|q,\downarrow\rangle$ with q being on-shell.

So in summary, the space of physical states (the cohomology of Q_B) can be seen as the space of states such that Q_B acting on any one of those states is zero and that the states are all on-shell in their momentum.

Conclusion

In this project, we explored the BRST formalism and demonstrated an application of the technique by studying the point particle in the context of string theory.

We showed that the BRST formalism builds upon the Faddeev-Popov method (see section 1), a method necessary for alleviating the mathematical challenge of over-integrating fields related by gauge transformations in gauge invariant theories.

The BRST formalism allows for the quantization of a gauge theory in a systematic and well-defined approach. Beyond this, the formalism also exhibits a symmetry that can be exploited to develop a powerful mathematical tool. The tool, namely the conserved charge, partitions the Hilbert space of allowed states in the gauge invariant theory into those which are physical and those which are non-physical. To understand how the partitioning occurs, it is important to consider the cohomology of the conserved charge (see section 1.5). In section 2.3, we demonstrated explicitly how the conserved charge partitions the Hilbert space of allowed states in the context of a relativistic point particle.

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